Optimal Pooling and Finite Mixture Distribution Combinations of Probabilistic Forecasts

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Combination of density forecasts

- Two-step procedures
- One-step procedures

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  1. estimate forecast models (parameter uncertainty)
     $F_1(\boldsymbol{\theta_1})$; $F_2(\boldsymbol{\theta_2})$
  2. and then, estimate combination weights (weight uncertainty)
     $y_{t+1} = \eta_1 F_1 + (1 - \eta_1) F_2 + \varepsilon_{t+1}$
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- One-step procedures
  jointly addresses both weight and parameter uncertainties estimating combination weights and forecasts' parameters simultaneously
  1. $y_{t+1} = \eta_1 F_1(\boldsymbol{\theta_1}) + (1 - \eta_1) F_2(\boldsymbol{\theta_2}) + \varepsilon_{t+1}$

Literature Review

- Two-step procedures
  - The optimal pooling: DeGroot and Mortera (1991), Stock and Watson (2004), Marcellino (2004), Wallis (2005), Genre et al. (2013); Kascha and Ravazzolo (2010), Gneiting et al. (2013).
  - Bayesian Model Averaging: Raftery et al. (1997), Raftery et al. (2005)Geweke and Whiteman (2006), Rossi and Sekhposyan (2014).
- One-step procedures
  - Finite mixture Models: LeSage and Magura (1992), Waggoner and Zha (2012), Billio et al. (2013), Del Negro et al. (2016), Mandel et al. (2016).
  - Infinite mixture models: Maheu and Yang (2016), Bassetti et al. (2018)

Motivation

Why a comparison between combination approaches?

- Are they the same?
- Which one is the most accurate?
- When one is more accurate than the other.

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Endow the DM with a tool to elicit not only the combination model but the family they belong to

In this paper

This paper proposes a comparison between

- Geweke and Amisano (2011) as two-step approach
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The comparison has carried out through:

Simulated data:

1. Baseline
2. Single-break
3. (Multiple-break)

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Simulated data:

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3. (Multiple-break)

Macroeconomic applications:

1. Small Dataset
2. Medium-Large Dataset

Two-step and One-step procedures

Two-step Procedure: the Optimal Pooling

1. A set of predictive distributions f_k for a vector of time series y_t
   $f_k = p(y_{t+1}|y_t^o, \boldsymbol{\theta}_k)$ Estimate $\boldsymbol{\theta}_k$ through OLS.

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2. are taken as given and combined according to:

$$g(y_t) = \sum_{k=1}^{K} \eta_k p(y_t; y_{t-1}^o, f_k)$$

where: $\sum_{k=1}^{K} \eta_k = 1$, $\eta_k \geq 0$, for $(k = 1, \ldots, K)$.

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Optimal weights η_k^*

are obtained by minimising the Kullback-Leiber Information Criterion (KLIC):

$$\eta_k^* = \underset{\eta}{\text{argmax}} \; \frac{1}{T} \sum_{t=1}^{T} \ln \; g(y_t)$$

Following Hall and Mitchell (2007) and Conflitti et al. (2015) details

One-step Procedure: Finite Mixture of Distributions Model

A standard Finite Mixture Distribution is used as limiting case of the Markov Switching Model (Waggoner and Zha, 2012):

$$p(y_t|y_{t-1}^0, \boldsymbol{\theta}_k) = \sum_{k=1}^{K} p(y_t|y_{t-1}^0, S_t = k, \boldsymbol{\theta}_k) Pr(S_t = k|y_{t-1}^0, \boldsymbol{\theta}_k)$$

$$p(y_t|y_{t-1}^0, \boldsymbol{\theta}_k) = \sum_{k=1}^{K} p(y_t|y_{t-1}^0, \boldsymbol{\theta}_k) \eta_k$$

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$$p(y_t|y_{t-1}^0, \boldsymbol{\theta}_k) = \sum_{k=1}^{K} p(y_t|y_{t-1}^0, \boldsymbol{\theta}_k)\eta_k$$

The weights are distributed according to

$$\eta_k \sim M\left(1, \left[\frac{p_1 f_N(y_t; \mu_{1,t}, \sigma_1^2)}{\sum_{k=1}^{K} p_k f_N(y_t; \mu_{k,t}, \sigma_k^2)}, \ldots, \frac{p_K f_N(y_t; \mu_{K,t}, \sigma_K^2)}{\sum_{k=1}^{K} p_k f_N(y_t; \mu_{k,t}, \sigma_k^2)}\right]\right)$$

where $p_k = (p_1, ..., p_K)$, $0 \leq p_k \leq 1$ and $\sum_{k=1}^{K} p_k = 1$.
Bayesian estimation with Gibbs sampling algorithm following Frühwirth-Schnatter (2006) details

Evaluation of Density Forecasts

- Logarithmic Scoring Rule: $LS(g(Y_t), A) = \log\,(g_i Y_t | A_i)$
- Continuous Ranked Probability Scores:

$$CRPS = \int_{-\infty}^{+\infty} (g(Y_{t+1}) - Y_{t+1})^2 dy$$

$$\text{TW-CRPS} = \int_{-\infty}^{+\infty} (g(z) - \mathbb{1}\{Y_{t+1} \leq z\})^2 w(z) dz$$

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- Probability Integral Transforms (PITs): the density forecast is called "calibrated" if z_t is uniform:

$$z_t = \int_{-\infty}^{Y_t} f_k(Y_t) d(Y_t)$$

Comparison with simulated data

No-break
Accuracy Loss: $(CRPS_{comb} - CRPS_{DGP})/CRPS_{DGP}$

- Data Generating Process (sample sizes $T = \{50, 200, 1000\}$)

$$y_t = 0.5 + 0.8y_{t-1} + \varepsilon_t \qquad \varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma_\varepsilon^2)$$

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- Predictive distributions f_k can be:

Nonnested case	Nested case
$f_1 : y_{T+h} \sim \mathcal{N}\big(\hat{\theta}_0 + \hat{\theta}_1 y_{T+h-1}, \hat{\sigma}^2_{\varepsilon_1}\big)$	$f_1 : y_{T+h} \sim \mathcal{N}\big(\hat{\theta}_0 + \hat{\theta}_1 y_{T+h-1}, \hat{\sigma}^2_{\varepsilon_1}\big)$
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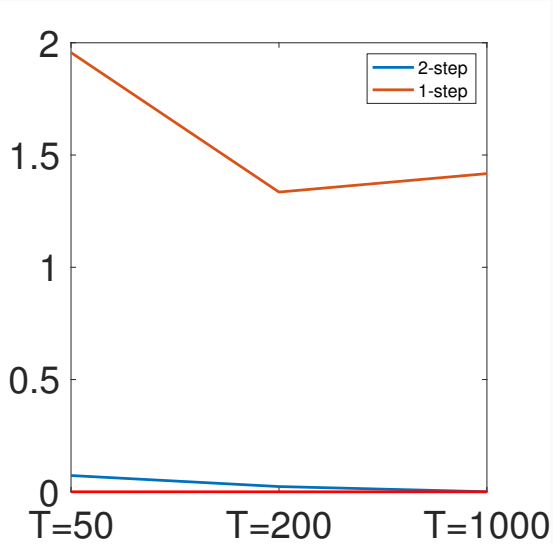

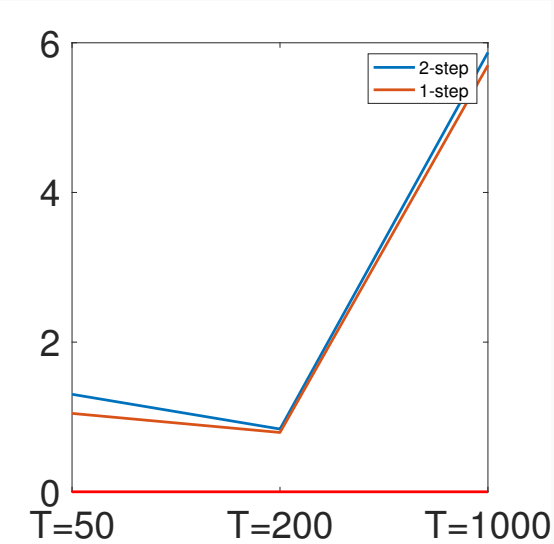

Single-Break Scenario: DGP Setups

- Data Generating Process

$$y_t = \Phi \boldsymbol{X}_t + \varepsilon_t \;\; \epsilon_t \overset{\text{iid}}{\sim} (0, \sigma_\varepsilon^2)$$

$$\boldsymbol{X}_t = \begin{bmatrix} 1 \\ y_{t-1} \\ x_{t-1} \end{bmatrix} \qquad \Phi = \begin{cases} [\phi_0 = 0.5 \quad \phi_1 = 0.8 \quad \phi_2 = 0] \;, & \text{if } t < T_b \\ [\phi_0 + d_0 \quad \phi_1 + d_1 \quad \phi_2 + d_2] \;, & \text{if } T_b \leq t \leq T. \end{cases}$$

$T = \{50, 200, 1000\}$ and $T_b = \{0.25, 0.50, 0.75, 0.95\}$ of T.

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f_1 and f_2 are combined treating the break as unknown

Single-break scenario: Types of break

exp. #	d_0	d_1	d_2	σ_ϵ^2	Comments
1	0	0	0	0.6	no breaks
2	-0.4	0	0	0.6	small break in the intercept
3	-0.6	0	0	0.6	large break in the intercept
4	0	-0.2	0	0.6	small break in AR(1) dynamics
5	0	-0.4	0	0.6	large break in AR(1) dynamics
6	0	0	0.5	0.6	Small break in exo. var. coefficient
7	0	0	1	0.6	Large break in exo. var. coefficient
8	0	-0.2	0.5	0.6	Breaks in AR(1) and exo. var. coefficients
9	0	0	0	2	Increase in post-break variance
10	0	0	0	0.3	Decrease in post-break variance

Single-break Nonnested case
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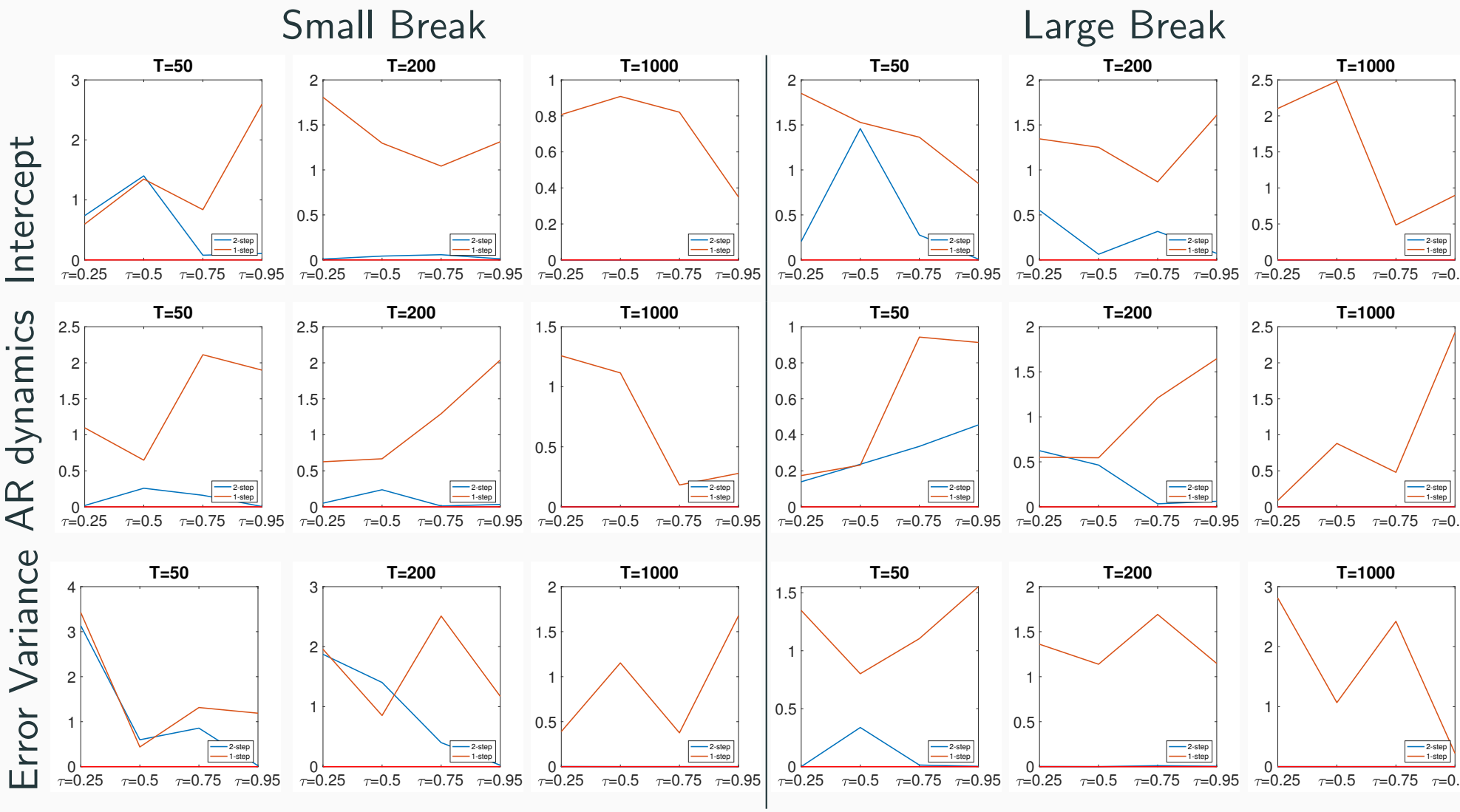

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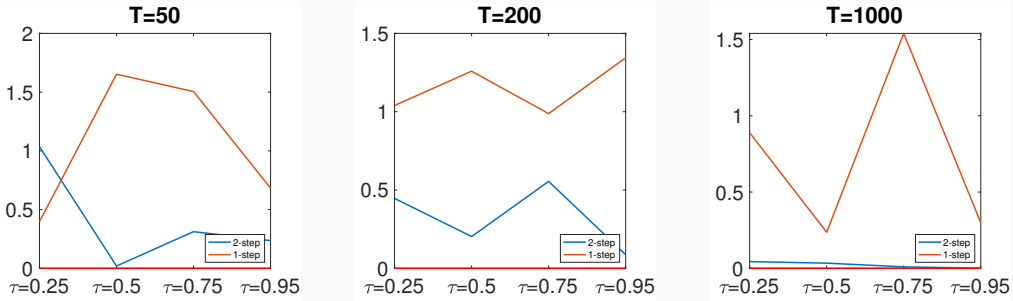
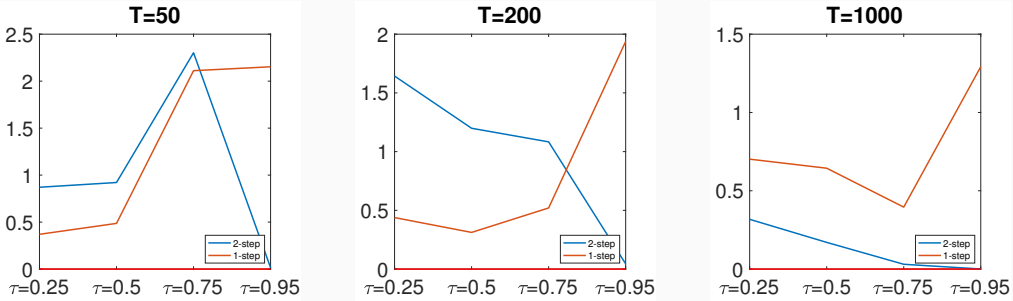
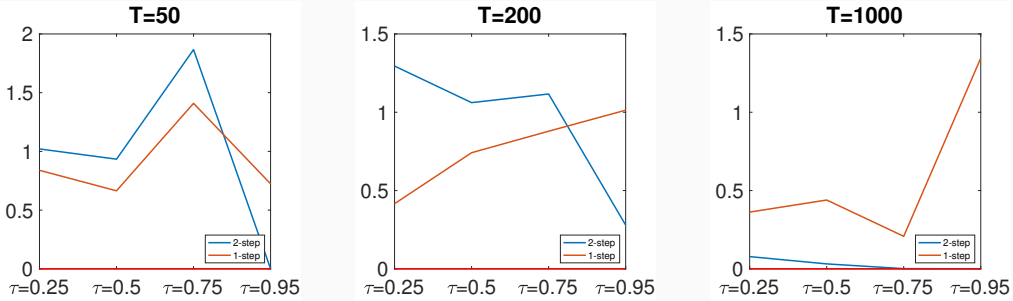
Small Break Exogenous Variable



Large Break Exogenous variable



Break in both AR and Exogenous var.



Single-break Nested case
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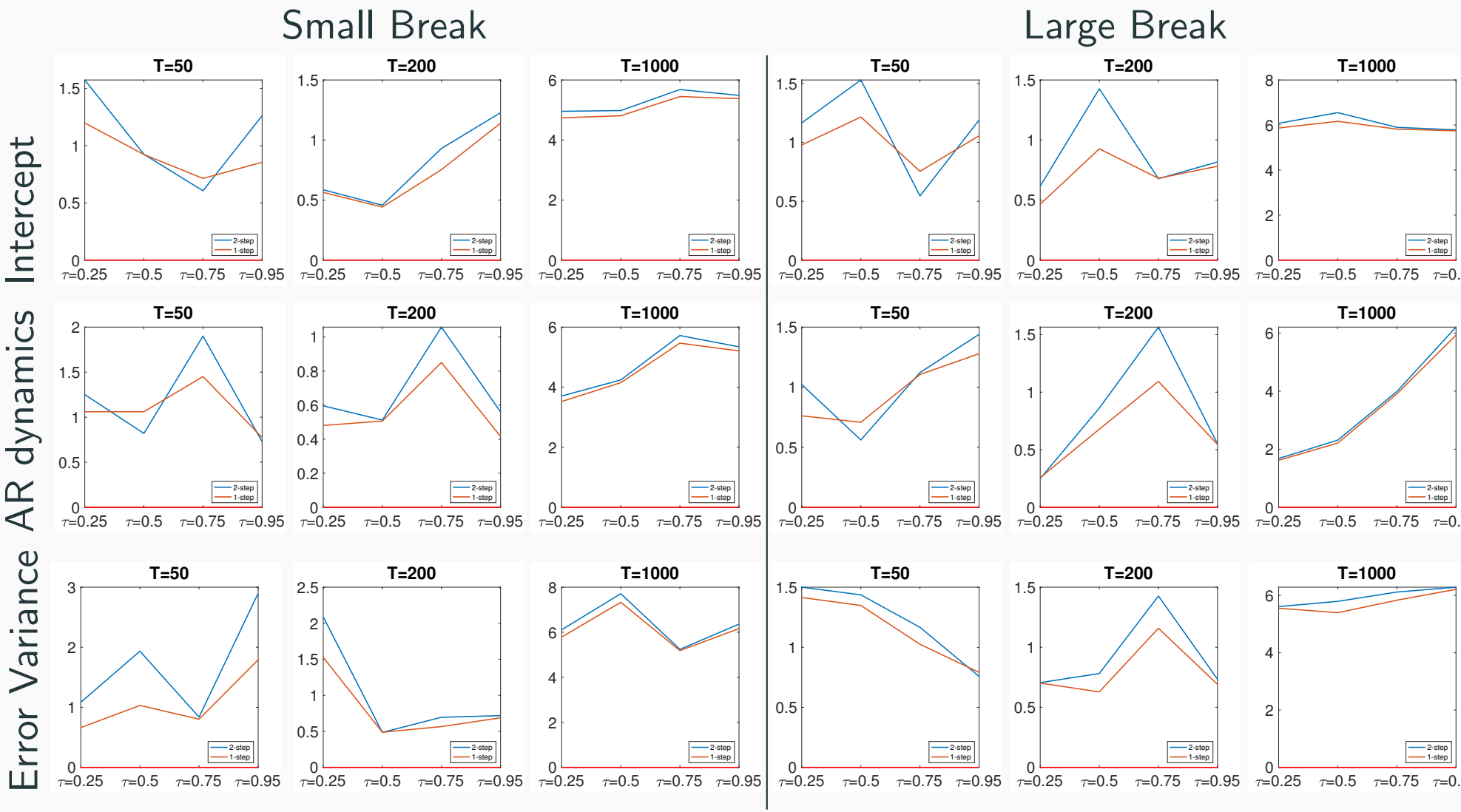

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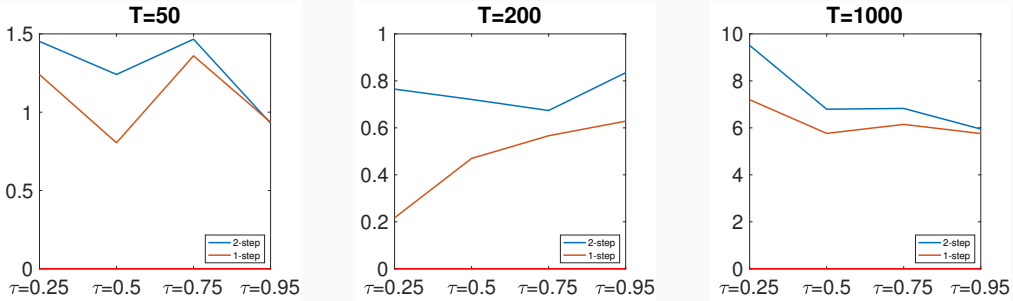

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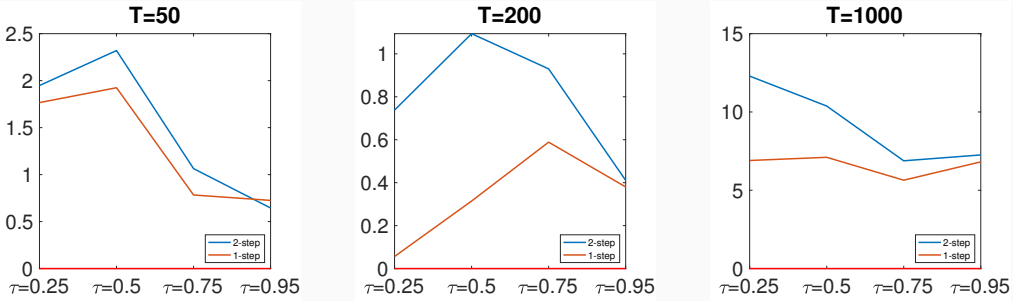
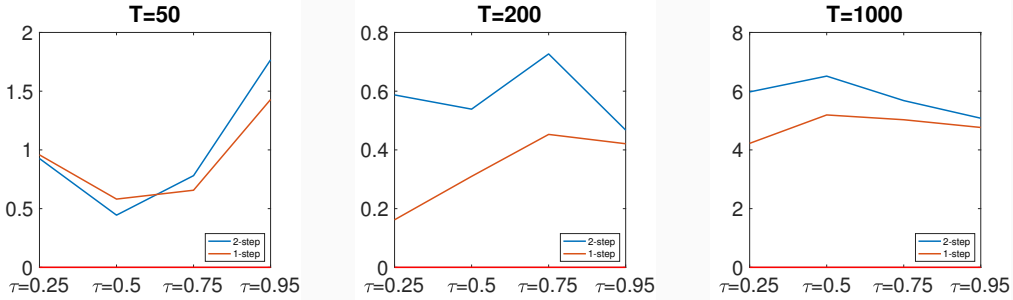


Multiple Breaks

Break in both AR and Exogenous var.



Empirical Applications

Forecasting GDP with industrial Production Index and Employment rate

Forecasting Real Output growth rate for US based on

- Industrial Production Index and Non-farm payroll employment
- In-sample forecast
- from 1964:Q1 to 2018:Q1
- One-step ahead rolling-window forecasts
- Using two datasets:
  1. Ex-post revised data (FRED database)
  2. Real-time data (Philadelphia Fed database)

Forecasting GDP with industrial Production Index and Employment rate

- Two predictive regressions:
  - $y_{t+1}^{IP} = \hat{\alpha}_{1,t} + \sum_{i=0}^{p} \hat{\beta}_{i,t} y_{t-i}^{o} + \sum_{j=0}^{Q} \hat{\delta}_{j,t} IP_{t-j} + \varepsilon_{1,t}$ and
  - $y_{t+1}^{EMP} = \hat{\alpha}_{2,t} + \sum_{i=0}^{p} \hat{\beta}_{i,t} y_{t-i}^{o} + \sum_{n=0}^{N} \hat{\gamma}_{n,t} EMP_{t-n} + \varepsilon_{2,t}$

  where $\varepsilon_{k,t} \sim N(0, \sigma_{k,t}^2)$.

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  where $\varepsilon_{k,t} \sim N(0, \sigma_{k,t}^2)$.
- Forecasts combined according to

$$p^c(y_{t+1}|y_t^o, \boldsymbol{\theta}_{k,t}) = \sum_{k=1}^{K=2} \hat{\eta}_{k,t} p(y_{t+1}|y_t^o, \boldsymbol{\theta}_{k,t})$$

Where $\Phi_{k,t} = \{\alpha_{k,t}, \delta_{j,t}, \beta_{i,t}, \gamma_{n,t}, \sigma_{k,t}^2, \eta_{k,t}\}$ are estimated according to the one-step and two-step procedures

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One-step and Two-step procedures compared with respect to:

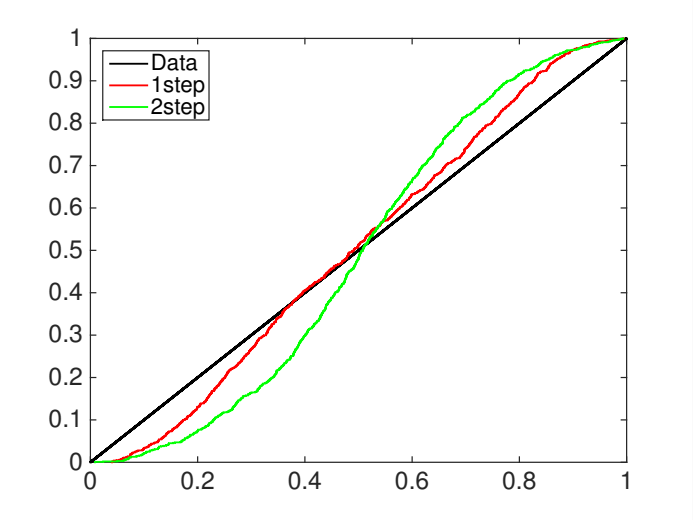
- a benchmark AR(1) model
- both density and point forecasts

Macroeconomic Application - Results

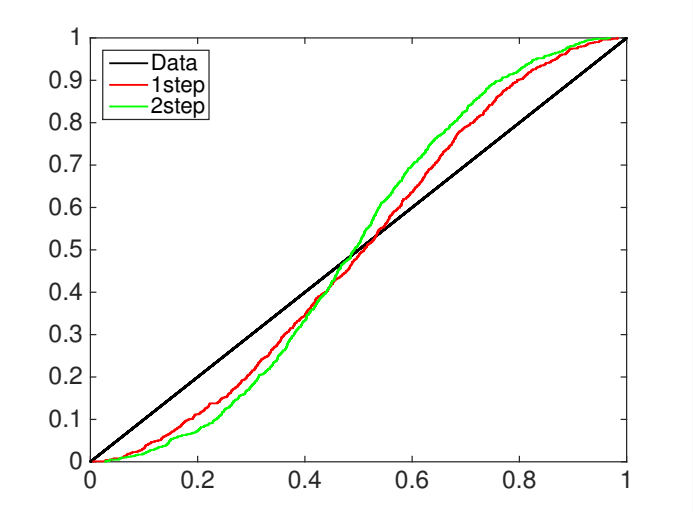
	Ex-Post Revised data		
	AR(1)	2-step	1-step
Log Score	-0.9818	-1.1765	-0.9134
		(0.000)	(0.000)
			(0.000)
CRPS	0.2289	0.4879	0.1331
		(0.000)	(0.000)
			(0.000)
TW-CRPS	0.2289	0.4880	0.1330
		(0.000)	(0.000)
			(0.000)
MSPE	0.2209	0.2215	0.2220
		(0.8236)	(0.3923)
			(0.8236)

	Real-Time data		
	AR(1)	2-step	1-step
Log Score	-0.9298	-0.9974	-0.9126
		(0.000)	(0.000)
			(0.000)
CRPS	0.1008	0.2916	0.0756
		(0.000)	(0.000)
			(0.000)
TW-CRPS	0.1008	0.2916	0.0760
		(0.000)	(0.000)
			(0.000)
MSPE	0.3629	0.3573	0.3914
		(0.3964)	(0.0362)
			(0.0521)

Ex-post revised data



Real-Time data



Forecasting US GDP and Inflation in a Large Macroeconomic Dataset

Forecasting Real Output growth and Inflation rate for US based on

- Replicate Rossi and Sekhposyan (2014)
- Subsample of Stock and Watson (2003) dataset details
- In-sample forecast
- from 1984:Q1 to 2010:Q3
- One-step ahead rolling-window forecasts of 40 observations

Forecasting US GDP and Inflation in a Large Macroeconomic Dataset

- K=32 forecasting models:

$$y_{t+1,k} = \beta_0 + \beta_1(L)X_{t,k} + \beta_{2,k}(L)y_t + \varepsilon_{t+1,k}$$

where $\varepsilon_{k,t} \sim N(0, \sigma^2_{k,t})$, $\beta_i(L) = \sum_{p=0}^{P} \beta_i L^i$

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$$p(y_{t+1}|y_t^o, \boldsymbol{\theta}_{k,t}) = \sum_{k=1}^{K} \eta_k p(y_{t+1,k}|y_t^o, \boldsymbol{\theta}_{k,t})$$

Where $\Phi_{k,t} = \{\beta_{i,t}, \sigma^2_{k,t}, \eta_{k,t}\}$ are estimated according to the one-step and two-step procedures algorithm

Forecasting US GDP and Inflation in a Large Macroeconomic Dataset

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One-step and Two-step procedures compared with respect to:

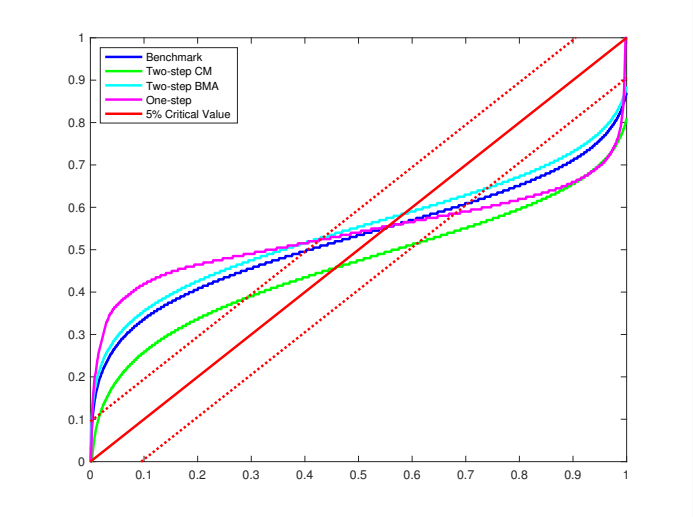
- a benchmark AR(1) model
- BMA pooling model as in Rossi and Sekhposyan (2014)

Large Macroeconomic Dataset Application - Results

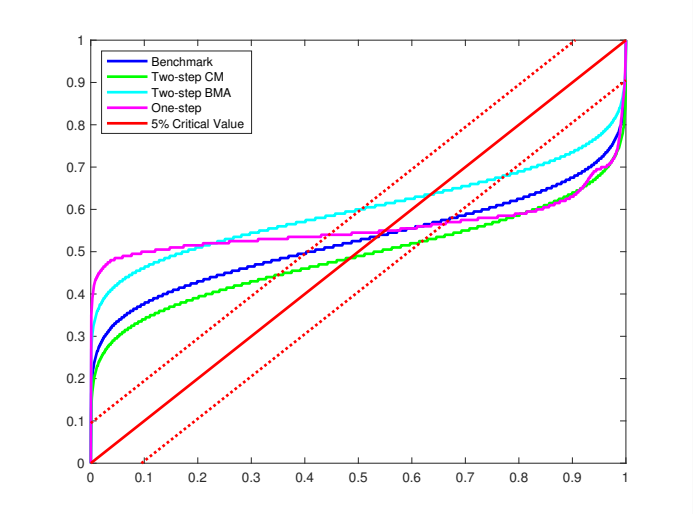
	AR(1)	2-stepCM	2-stepBMA	1-step
LogScore	-0.9258	-0.9376	-0.9247	-0.9300
		(0)	(0.3003)	(0)
			(0)	(0.0008)
				(0)
CRPS	0.2312	0.2406	0.2310	0.2374
		(0)	(0.6521)	(0)
			(0.0006)	(0.0966)
				(0)
TW-CRPS	0.2313	0.2408	0.2311	0.2372
		(0)	(0.6239)	(0)
			(0.0006)	(0.0737)
				(0)

	AR(1)	2-step CM	2-step BMA	1-step
Log Score	-0.9220	-0.9278	-0.9212	-1.1458
		(0.0062)	(0.0008)	(0)
			(0.0025)	(0)
				(0)
CRPS	0.2427	0.2467	0.2426	0.5211
		(0.0118)	(0.9014)	(0)
			(0.0113)	(0)
				(0)
TW-CRPS	0.2429	0.2466	0.2425	0.5211
		(0.0178)	(0.0015)	(0)
			(0.0103)	(0)
				(0)

GDP



Inflation



Conclusions

To elicit the combination approaches, the DM has to account for:

- Individual models specification:
  - Nested in favour of one-step approach
  - Nonnested in favour of two-step approach
- Number of models to combine:
  - Two-step is not affected by the number of individual models
  - Carefully accounting for large dataset estimating the one-step approach.

Appendix

Conflitti et al. (2015)

$$\Phi(\omega_j^*) = \frac{1}{T}\sum_{t=1}^{T} \ln\, g(Y_t) \tag{1}$$

where $\omega_{\hat{OPT}}$ maximises $\Phi(\omega_j^*)$ subject to the constraints $\omega_j \geq 0$ and $\sum_{j=1}^{m} \omega_j = 1$. Let us define the $T \times J$ matrix $\hat{G}$ composed by nonnegative elements $\hat{G}_{tj} = \hat{g}_t(Y_t)$. Then equation 1 can be rewritten as:

$$\Phi(\omega_j^*) = \frac{1}{T}\sum_{t=1}^{T} \ln\, (\hat{G}\omega_j). \tag{2}$$

Let as introduce the following Lagrange multiplier λ to take into account the constraints of the weights:

$$\Phi(\omega_j^*) = \frac{1}{T}\sum_{t=1}^{T} \ln\, (\hat{G}\omega_j) - \lambda \sum_{j=1}^{m} \omega_j. \tag{3}$$

Following Conflitti et al. (2015), we introduce a "surrogate" cost function depending on a vector of arbitrary weights a_j, such that:

$$\Psi_\lambda(\omega_j, a_j) = \frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{m}\frac{\hat{G}_{jt}a_j}{\sum_{j=1}^{m}\hat{G}_{jt}a_j}\ln\left(\frac{\omega_j}{a_j}\sum_{j=1}^{m}\hat{G}_{jt}a_j\right) - \lambda\sum_{j=1}^{m}\omega_j. \tag{4}$$

Let us define the following algorithm for k numbers of iterations:

$$\omega_{j,\lambda}^{k+1} = \underset{\omega}{\operatorname{argmax}}\ \Psi_\lambda(\omega_j, \omega_{j,\lambda}^{k}) \tag{5}$$

Rewriting last equation in terms of ω_j^k, the iterative algorithm becomes:

$$\omega_j^{k+1} = \omega_j^k\frac{1}{T}\sum_{t=1}^{T}\frac{\hat{G}_{jt}}{\sum_{j=1}^{m}\hat{G}_{jt}\omega_j^k}. \tag{6}$$

The nonnegative constrain is satisfied imposing positive weights that sum to one as initial values (i.e. $\omega_j^0 = 1/m$). The iterates are expected to converge to the maximiser $\omega_{\hat{OPT}}$ due to the monotonicity of the cost function in (4) and the constraints. The algorithm has also a stop criterion based on negligible difference between two successive iterates. back

Description of Data Series

Label	Trans	Period	Name	Description	Source
			Asset Prices		
rovnght@us	level	59:M1-10:M9	FEDFUNDS	Int. Rate: Fed Funds (Effective)	F
rtbill@us	level	59:M1-10:M9	TB3MS	Int. Rate: 3-Mn Tr. Bill, Sec Mkt Rate	F
rbnds@us	level	59:M1-10:M9	GS1	Int. Rate: US Tr. Const. Mat., 1-Yr	F
rbndm@us	level	59:M1-10:M9	GS5	Int. Rate: US Tr. Const. Mat., 5-Yr	F
rbndl@us	level	59:M1-10:M9	GS10	Int. Rate: US Tr. Const. Mat., 10-Yr	F
stockp@us	Δln	59:M1-10:M9	SP500	US Share Prices: S&P 500	F
exrate@us	Δln	73:M1-10:M9	111	NEER	I
			Real Activity		
rgdp@us	Δln	59:Q1-10:Q3	GDPC12	Real GDP, sa	F
ip@us	Δln	59:M1-10:M9	INDPRO	Industrial Production Index, sa	F
capu@us	level	59:M1-10:M9	CUMFNS	Capacity Utilization Rate: Man., sa	F
emp@	Δln	59:M1-10:M9	CE16OV	Civilian Employment: thsnds,sa	F
unemp@us	level	59:M1-10:M9	UNRATE	Civilian Unemployment,sa	F
			Wages and Prices		
pgdp@us	Δln	59:Q1-10:Q3	GDPDEF	GDP Deflator, sa	F
cpi@us	Δln	59:M1-10:M9	CPIAUCSL	CPI: Urban, All items, sa	F
ppi@us	Δln	59:M1-10:M9	PPIACO	Producer Price Index, nsa	F
earn@us	Δln	59:M1-10:M9	AHEMAN	Hourly Earnings: Man., nsa	F
			Money		
mon0@us	Δln	59:M1-10:M9	AMBSL	Monetary Base, sa	I
mon1@us	Δln	59:M1-10:M9	M1SL	Money: M1, sa	I
mon2@us	Δln	59:M1-10:M9	M2SL	Money: M2, sa	I
mon3@us	Δln	59:M1-06:M2	M3SL	Money: M3, sa	I

back

One-step procedure - Bayesian Inference

Bayesian inference technique of MCMC estimation using data augmentation and Gibbs sampling.
Posterior Distributions are:

- Combination Weights η_k: $p(\eta_k|\boldsymbol{S}) \sim \mathcal{D}(e_1(\boldsymbol{S}), \ldots, e_K(\boldsymbol{S}))$
- Autoregressive parameters ϕ_k: $p(\phi_k|\boldsymbol{\sigma}_k^2, y_t^o, \boldsymbol{S}) \sim \mathcal{N}(a_k, A_k)$
  where $A_k = (A_0^{-1} + \frac{1}{\sigma_k^2}\boldsymbol{z}_k'\boldsymbol{z}_k)^{-1} \quad ; \quad a_k = A_k(A_0^{-1}a_0 + \frac{1}{\sigma_k^2}\boldsymbol{z}_k'y_k)$
- Variances σ_k^2: $p(\sigma_k|\boldsymbol{\theta}_k, y_t^o, \boldsymbol{S}^{m-1}) \sim \mathcal{G}^{-1}(c_k, C_k)$
  where $c_k = c_0 + \frac{N_k}{2}, \qquad C_k = C_0 + \frac{1}{2}\epsilon_k'\epsilon_k$ and $\epsilon_k = y_t^o - Z_k\phi_k$.

A common choice of prior takes the form:

$$p(\boldsymbol{\theta}_k) = \mathcal{D}(e_0, e_0|\boldsymbol{S}) \prod_{k=1}^{2} \mathcal{N}(\phi_k|a_0, A_0)\mathcal{G}^{-1}(\sigma_k^2|c_0, C_0).$$

One-step procedure - Bayesian Inference

Dealing with Label Switching

$$p(y_t|\boldsymbol{\theta}_k, \eta_k) = \sum_{k=1}^{K} \eta_k f_N(y_t|\boldsymbol{\theta}_k) = \sum_{k=1}^{K} \eta_{\rho(k)} f_N(y_t|\boldsymbol{\theta}_{\rho(k)})$$

where ρ is an arbitrary permutation of $\{1, \ldots, K\}$.

back.

3. Multi-break Scenario

$$y_t = \Phi \boldsymbol{X}_t + \varepsilon_t \qquad \epsilon_t \overset{\text{iid}}{\sim} (0, \sigma_\varepsilon^2)$$

$$\boldsymbol{X}_t = \begin{bmatrix} 1 \\ y_{t-1} \\ x_{t-1} \end{bmatrix} \qquad \Phi = \begin{cases} [\phi_0 = 0.5 \quad \phi_1 = 0.8 \quad \phi_2 = 0] \ , & \text{if } t < T_b \\ [\phi_0 + d_0 \quad \phi_1 + d_1 \quad \phi_2 + d_2] \ , & \text{if } T_b \leq t \leq T_c \\ [\phi_0 + d_0^* \quad \phi_1 + d_1^* \quad \phi_2 + d_2^*] \ , & \text{if } T_c \leq t \leq T \end{cases} \tag{7}$$

$T = \{50, 200, 1000\}$, $T_b = 0.25$, $T_c = 0.75$ of T.

exp #	d_0	d_1	d_2	d_0^*	d_1^*	d_2^*	σ_ϵ^2	Comments
11	-0.2	0	0	0	0	0	0.6	mean reversion in intercept
12	-0.2	0	0	-0.4	0	0	0.6	decreasing trend in the intercept
13	0	-0.2	0	0	0	0	0.6	mean reversion in AR dynamics
14	0	-0.2	0	0	-0.4	0	0.6	decreasing trend in AR dynamics
15	0	0.2	0	0	0.4	0	0.6	increasing trend in AR dynamics
16	0	0	1	0	0	0	0.6	mean reversion in predictor coefficient
17	0	0	1	0	0	2	0.6	trending break in predictor coefficient
18	0	0	0	0	0	0	2	increase in post-break variance
19	0	0	0	0	0	0	0.3	decrease in post-break variance

2. Combination

- Individual Models

Nonnested case:	Nested case:
$f_1 : y_{T+h} \sim \mathcal{N}\big(\hat{\theta}_0 + \hat{\theta}_1 y_{T+h-1}, \hat{\sigma}^2_{\varepsilon_1}\big)$	$f_1 : y_{T+h} \sim \mathcal{N}\big(\hat{\theta}_0 + \hat{\theta}_1 y_{T+h-1}, \hat{\sigma}^2_{\varepsilon_1}\big)$
$f_2 : y_{T+h} \sim \mathcal{N}\big(\hat{\beta}_0 + \hat{\beta}_1 x_{T+h-1}, \hat{\sigma}^2_{\varepsilon_2}\big)$	$f_2 : y_{T+h} \sim \mathcal{N}\big(\hat{\theta}_0 + \hat{\theta}_1 y_{T+h-1} + \hat{\beta}_1 x_{T+h-1}, \hat{\sigma}^2_{\varepsilon_2}\big)$
$f_3 : y_{T+h} \sim \mathcal{N}\big(\hat{\beta}_0 + \hat{\beta}_1 x_{T+h-1}, \hat{\sigma}^2_{\varepsilon_3}\big)$	$f_3 : y_{T+h} \sim \mathcal{N}\big(\hat{\theta}_0 + \hat{\theta}_1 y_{T+h-1} + \hat{\beta}_1 x_{T+h-1}, \hat{\sigma}^2_{\varepsilon_3}\big)$

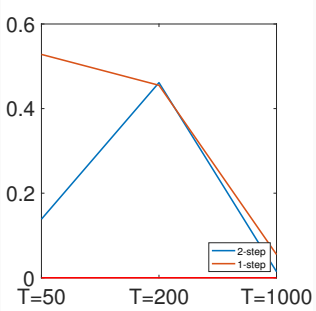
- $\hat{\theta}_0, \hat{\beta}_0, \hat{\theta}_1, \hat{\beta}_1, \hat{\sigma}^2_{\varepsilon_1}, \hat{\sigma}^2_{\varepsilon_2}$ are estimated by Two-step and One-step approach,
- f_1 and f_2 are combined treating the break as unknown.

back

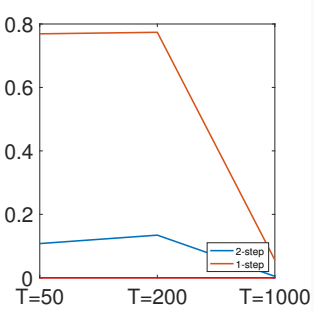
3. Multi-break Results

Nonnested Case

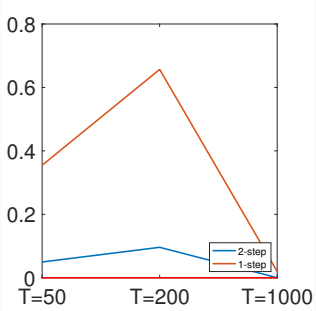
Exp.11



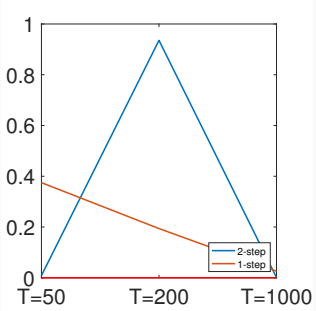
Exp.12



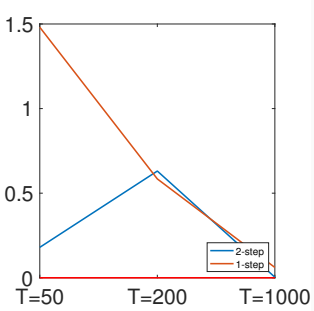
Exp.13



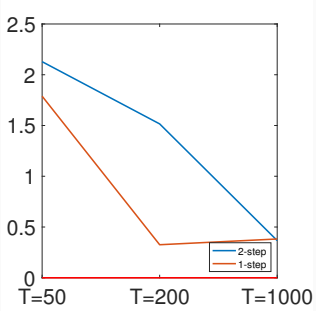
Exp.14



Exp.15



Exp.16



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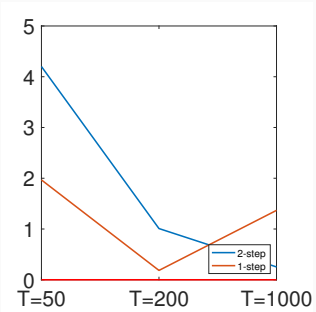
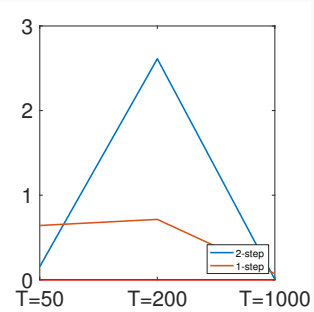
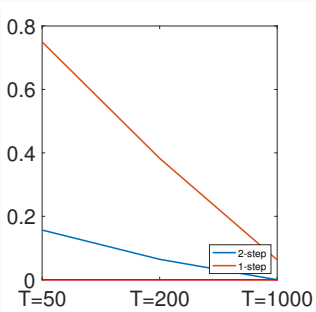


Exp.18

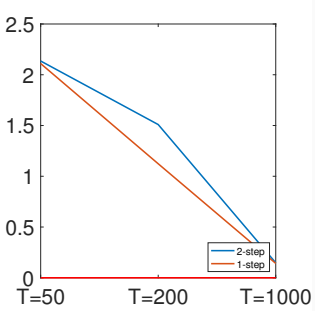


Exp.19

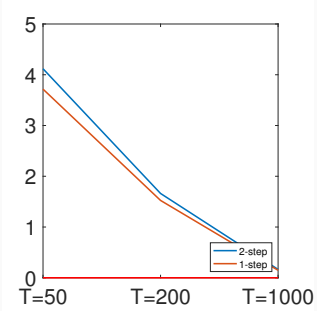


Nested case

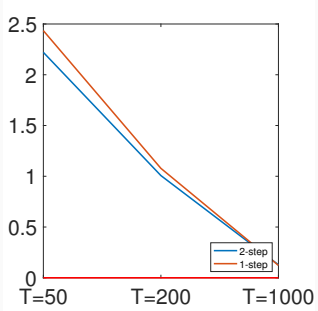
Exp.11



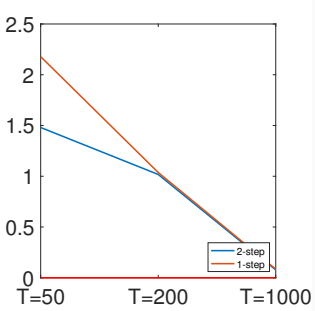
Exp.12



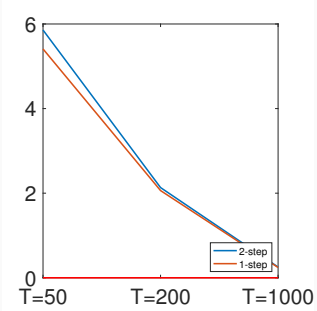
Exp.13



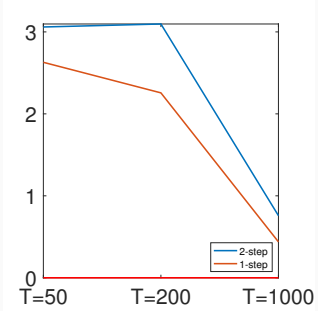
Exp.14



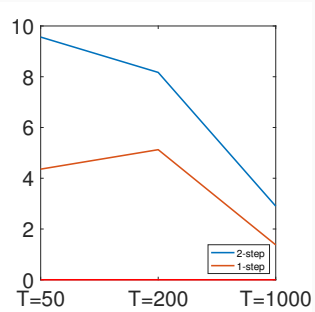
Exp.15



Exp.16



Exp.17



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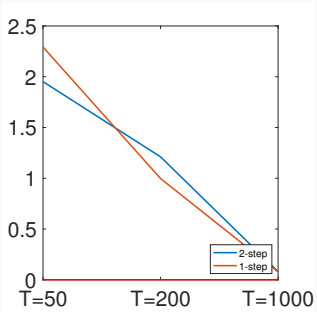

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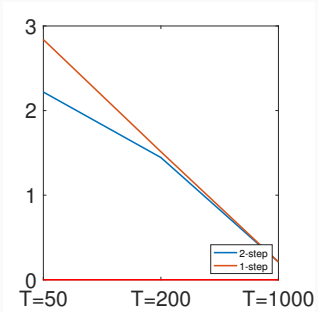

Algorithm for large dataset Application

Start from some initial values of $\boldsymbol{S}^0$ and repeat the following steps M times after a burn-in period long M_0.

1. parameter simulation conditional on the allocation $\boldsymbol{S}^{m-1}$ (as in algorithm (1)):
   - 1.1 Sample η_k from the conditional Dirichlet posterior $p(\eta_k|\boldsymbol{S})$ as in algorithm (1);
   - 1.2 Sample each regression coefficient $\phi = (\phi_{1,0}, \phi_{1,1}, \phi_{2,0}, \phi_{2,1}, \phi_{2,2})$ jointly from the posterior distribution $p(\phi|\sigma_k^2, y_t^o, \boldsymbol{S}^{m-1}) \sim \mathcal{N}(a_k, A_k)$ as in algorithm (1);
   - 1.3 Sample the random hyperparameter C_0 from $p(C_0|\boldsymbol{S}^{m-1}, \boldsymbol{x}_i\phi_k, \sigma_k^2, y_t^o \sim \mathcal{G}(g_N, G_N)$;
   - 1.4 Sample each variance σ_k^2 from the posterior distribution $\sigma_k|\phi, y_t^o, \boldsymbol{S}^{m-1} \sim \mathcal{G}^{-1}(c_k, C_k)$ Where $c_k = c_0 + \frac{N_k}{2}$ and $C_k = C_0 + \frac{1}{2}\epsilon_k'\epsilon$
2. Classification of each observation y_t conditional on $\boldsymbol{\theta}_k$: sample each element of S_i of $\boldsymbol{S}^m$ from the conditional posterior $p(S_t|\phi, \sigma_{\epsilon,k}^2, y_t^o)$ given by:

$$Pr(S_i = k|\phi, \sigma_k^2, y_t^o) \propto \eta_k f_N(y_t^o; \boldsymbol{x}_i\phi_k, \sigma_k^2)$$

The posterior density estimated from the MCMC draws is:

$$p(y_{t+1}|y_t^o, \boldsymbol{\theta}_k) = \frac{1}{M + M_0} \sum_{m=1}^{M} \left(\sum_{k=1}^{K} \eta_k^m p(y_{t+1}|\theta_{k,t+1}^m) \right)$$

back

Preliminary Conclusions

1. Combination of two AR models suggests the superiority of one-step procedure with sufficiently large sample size;
2. In presence of breaks:
   - 2.1 The specification of individual models matters
     - Nonnested case: Two-step is more accurate;
     - Nested case: One-step is more accurate;
   - 2.2 The type of break matters:
     - One-step performs better when the break regards exogenous variables.
   - 2.3 The sample size matters:
     - In the nonnested case, two-step becomes more accurate than one-step
     - In the nested case, one-step becomes more accurate than two-step
   - 2.4 No clear information from the timing of the break.