Further Results on Identification of Structural VAR Models

Andrzej Kocięcki
National Bank of Poland
e–mail: andrzej.kociecki@nbp.pl

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Abstract: We provide some generalization and clarification of the identification conditions for Structural VAR (SVAR) models given in Rubio–Ramírez et al (2010). In particular we show that their basic sufficient condition is also necessary. In addition we give necessary and sufficient conditions for identification almost everywhere in SVAR under homogenous restrictions irrespective of whether the model is exactly identified or over–identified. The modification of the order condition is also suggested.

I. Introduction

In a recent important article by Rubio–Ramírez et al (2010), to be referred to as RWZ, the authors gave the first serious and comprehensive treatment of the identification problem in Structural VAR (SVAR) models. They provide sufficient condition for identification in over–identified SVAR and necessary and sufficient condition for identification in exactly identified models. In the latter case RWZ also proved that if the condition holds at any parameter point then the identification is attained almost everywhere. Building on RWZ the contribution of the present paper is 1) the proof that the sufficient condition in RWZ is also necessary, 2) the necessary and sufficient conditions for identification almost everywhere in over–identified models 3) slight modification of the necessary condition for global identification.

II. Model Setup

As far as the notation is concerned we follow RWZ as close as possible hence we will be very concise (see RWZ for clarifying comments). We work with the following SVAR model

$$y_t' A_0 = y_{t-1}' A_1 + \cdots + y_{t-p}' A_p + c + \varepsilon_t' ; \qquad \text{for } t = 1, \ldots, T \tag{1}$$

where $A_0 : (n \times n)$ is the nonsingular matrix of contemporaneous relations between the data $y_t : (n \times 1)$, $A_i : (n \times n)$, $c : (1 \times n)$ is a vector of constants and $\varepsilon_t \mid y_{t-1}, y_{t-2}, \ldots \sim N(0_{n\times 1}, \mathrm{I}_n)$. Let us define $A_+' = [A_1' \ldots A_p' \; c']$. A_+ is an $m \times n$ matrix, where $m = np + 1$. Further let O_n denote the space of $(n \times n)$ orthogonal matrices i.e. $O_n = \{P \in \mathbb{R}^{n\times n} \mid P'P = PP' = \mathrm{I}_n\}$.

The set of all structural parameters is denoted as $\mathbb{P}^S$. Let $(A_0, A_+) \in \mathbb{P}^S$ be any parameter point. We say that the model is globally identified at (A_0, A_+) if for any other $(\bar{A}_0, \bar{A}_+) \neq (A_0, A_+)$ the induced probability measures $\mathcal{P}_{A_0, A_+}$ and $\mathcal{P}_{\bar{A}_0, \bar{A}_+}$ are different. The general SVAR model (1) without any restrictions will not be identified. Following RWZ we consider only homogenous restrictions that may be written

$$Q_j f(A_0, A_+) e_j = 0 \text{, for } j = 1, \ldots, n \tag{2}$$

where $f(A_0, A_+) : (k \times n)$ for some $k > 0$, e_j is the $j-$th column of I_n. The minor modification in comparison to RWZ is that $Q_j : (q_j \times k)$ with $rank(Q_j) = q_j$ (Q_j in RWZ comprises our "Q_j" and $(k - q_j) \times k$ block of zeros). The domain of $f(\cdot)$ is some $U \subset \mathbb{P}^S$ and the transformation $f(\cdot)$ must be admissible (for any $P \in O_n$ and $(A_0, A_+) \in U$, $f(A_0 P, A_+ P) = f(A_0, A_+) P$) and fulfill some regularity conditions (see RWZ). The great insight of RWZ is that all linear and most of the nonlinear restrictions met in practice may be cast in the form (2). As noted by RWZ it is important to permute the columns of $f(A_0, A_+)$ so as it holds that $q_1 \geq q_2 \geq \ldots \geq q_n$. Let us denote the $j-$th column of such a permuted $f(A_0, A_+)$ as f_j and let us signify $f_{[j]} = [f_j \; f_{j+1} \ldots f_n]$. You should be aware that f_j is the implicit function of A_0, A_+ and we use this notation for economical reasons.

Since restrictions (2) are homogenous they may at best identify SVAR model only up to arbitrary sign of each equation. Some normalization is needed in addition to (2) to attain the global identification. To distinguish the identification up to arbitrary sign of each equation from the concept of global identification we term the former as the regional identification (which is something between being local and global).

Definition 1: *The SVAR model is regionally identified at* $(A_0, A_+) \subset U$ *if and only if (ifif)*[1] $\{P \in O_n \mid (A_0P, A_+P) \in U\ ,\ \ Q_j f(A_0P, A_+P)e_j = 0\ ,\ \text{for}\ \ j = 1,\ldots,n\ \} = D$, *where* $D = \{diag(\delta_1,\ldots,\delta_n) \mid \delta_i = \pm 1\}$.

Properly speaking definition 1 is the lemma that states that the basic identification definition (mentioned earlier) when all the restrictions are in a form (2) (which lacks the normalization) is equivalent to definition 1. Without the normalization all we can have is the identification up to each equation's sign (i.e. regional identification).

Following definition 3 in RWZ, let $N \subset \mathbb{P}^S$ be a normalization rule. Then slight modification of definition 1 leads to the underlying global identification (as understood by RWZ)

Definition 2: *The SVAR model is globally identified at* $(A_0, A_+) \subset U$ *ifif* $\{P \in O_n \mid (A_0P, A_+P) \in U \cap N\ ,\ \ Q_j f(A_0P, A_+P)e_j = 0\ ,\ \text{for}\ \ j = 1,\ldots,n\ \} = \{d\}$, *where* $d \in D$.

What is important in definition 2 is that $d \in D$ must be unique i.e. $\{d\}$ denotes a singleton.

In contrast to RWZ we concentrate on regional identification. But it should be clear that regional identification plus "reasonable" normalization amounts to achieving global identification. For criteria of "reasonableness" the reader is referred to Waggoner and Zha (2003) and Hamilton et al. (2007). Hence in what follows, the regional identification is the synonym for the global identification.

[1] We use "ifif" instead of the usual "iff" following suggestion of I.J. Good. He used to say that "ifif" is at least pronounceable neologism ("iff" is the barbarism).

III. The Necessary and Sufficient Condition for Regional Identification

The aim of this section is to demonstrate that the sufficient condition for identification given in theorem 1 in RWZ is also necessary. We begin with a derivation of the necessary and sufficient condition for regional identification.

Proposition 1 (rank condition): *Let* $q_1 \geq q_2 \geq \ldots \geq q_n$. *Necessary and sufficient condition for SVAR to be regionally identified at* (A_0, A_+) *is that* $rank(Q_j f_{[j+1]}) = n - j$, *for all* $j = 1, \ldots, n-1$.

Proof: see appendix 1.

Since $Q_j f_{[j+1]}$ is a $q_j \times (n-j)$ matrix, using proposition 1 we can state the refinement of the necessary (i.e. order) condition for global identification in SVAR models

Corollary 1 (order condition): *Let* $q_1 \geq q_2 \geq \ldots \geq q_n$. *Then the necessary condition for global identification of SVAR is* $q_j \geq n - j$, *for* $j = 1, \ldots, n$.

The order condition in corollary slightly differs from the common necessary condition that requires that the total number of restrictions must be greater than $\frac{1}{2}n(n-1)$. However note that $q_j \geq n - j$, for $j = 1, \ldots, n$, implies $q = \sum_{j=1}^{n} q_j \geq \frac{1}{2}n(n-1)$. Thus our necessary condition is stronger than the common one. The corollary makes it explicit that what really matters is not only the number of restrictions but also its distribution over all equations. This should not be confused with the similar statements in RWZ in the context of exactly identified models.

In RWZ the crucial role plays the following matrix, which in our notation reads

$$\underset{(k+j)\times n}{M_j(f(A_0, A_+))} = \begin{bmatrix} Q_j f(A_0, A_+) \\ 0_{(k-q_j)\times n} \\ [\mathrm{I}_j \ 0_{j\times(n-j)}] \end{bmatrix} \quad (3)$$

Their theorem 1 states that if $rank(M_j(f(A_0, A_+))) = n$; for $j = 1, \ldots, n$, then the SVAR is regionally identified at (A_0, A_+). Although they emphasize that this condition is only sufficient for regional identification we can easily prove that in fact the condition is also necessary.

Proposition 2: *Let* $q_1 \geq q_2 \geq \ldots \geq q_n$. *Then* $rank(M_j(f(A_0, A_+))) = n$; $\forall\ j = 1,\ldots,n$, *is necessary and sufficient for regional identification at* (A_0, A_+).

Proof:

$$rank(M_j(f(A_0, A_+))) = rank\begin{bmatrix} Q_j f(A_0, A_+) \\ [\mathrm{I}_j\ 0_{j\times(n-j)}] \end{bmatrix} = rank\begin{bmatrix} \underbrace{Q_j f_1 \ldots Q_j f_j}_{q_j \times j} & Q_j f_{[j+1]} \\ \mathrm{I}_j & 0_{j\times(n-j)} \end{bmatrix} = rank(Q_j f_{[j+1]}) + j$$

The proof of the last equality may be found in e.g. Abadir and Magnus (2005), exercise 5.46. By proposition 1, $rank(Q_j f_{[j+1]}) = n - j$; for $j = 1,\ldots,n-1$, is necessary and sufficient for regional identification at (A_0, A_+). But for $j = 1,\ldots,n-1$, $rank(Q_j f_{[j+1]}) = n - j$ if and only if $rank(M_j(f(A_0, A_+))) = n$. To complete the proof note that in fact $rank(M_n(f(A_0, A_+))) = n$ for all A_0, A_+.

Hence you may use our proposition 1 or equivalently theorem 1 in RWZ to check out the regional identification.

IV. Identification Almost Everywhere

Since the unknown "true" parameter point is always unknown, it is important to have criteria to find out whether identification holds for all or almost all parameter points in the parameter space. Unfortunately the uniform identification (i.e. "for all") characterizes only special SVAR models (e.g. with recursive identifying scheme on A_0). Otherwise we can only hope for the identification almost everywhere [Lebesgue]. Important contribution of RWZ was realizing us that if SVAR under restrictions (2) is identified at arbitrary parameter point then the model is identified for almost all parameter points (theorem 3, RWZ). However we note that the special case of such a result, when there are only linear restrictions, has been well known for many years. See e.g. Koopmans (1950), pp. 82–83, Fisher (1966), pp. 44–45, and Johansen (1995), theorem 2. The case of the nonlinear restrictions, which however may be cast in the form (2), does not change anything.

It is useful to distinguish between the cases when all restrictions are exclusive (i.e. "zeros") and at least one restriction is not a "zero". There are two reasons for that. First, in practice most SVAR applications fall in the first category. Second, exclusive restrictions allow for the easier theoretical treatment and, as we show, result in a more intuitive and more easily checkable conditions for identification almost everywhere (in comparison to general linear restrictions).

First we deal with exclusive restrictions. To accomplish it we need basic notions from combinatorics. Let X be any matrix. Define the line to be either a row or a column of X. The term rank of X, to be denoted as $\rho(X)$, is the maximal number of non–zero elements of X with no two non–zero elements on a line. In our context the non–zero elements are those that are not restricted to zero by identifying restrictions. For example

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 \\ 0 & 0 & x_{23} & 0 \\ 0 & 0 & 0 & x_{34} \\ x_{41} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{54} \end{bmatrix} \tag{4}$$

has the term rank 4 (e.g. take $\{x_{41}, x_{12}, x_{23}, x_{54}\}$). But if x_{41} were also 0 then $\rho(X) = 3$. By the classic König's theorem, $\rho(X)$ is equal to the minimal number of lines that contain all the non–zero elements in X (see e.g. Ryser (1963) pp. 55–56). For example, if x_{41} were 0 in (4) then the first row, the third column and the fourth column would comprise all the non–zero elements in X.

Proposition 3: *Suppose that all restrictions are exclusive. Let* $q_1 \geq q_2 \geq \ldots \geq q_n$. *Then* $rank(Q_j f_{[j+1]}) = n - j$, *for all* $j = 1,\ldots,n-1$, *almost everywhere ifif the term rank* $\rho(Q_j f_{[j+1]}) = n - j$, *for all* $j = 1,\ldots,n-1$.

Proof: See appendix 2.

Equivalently instead of $\rho(Q_j f_{[j+1]})$ one may count the minimum number of lines in $Q_j f_{[j+1]}$ that contain all the non–zero elements of $Q_j f_{[j+1]}$ (by König's theorem).

The case of the general linear restrictions requires slightly different approach[2]. One may say that the importance of proposition 3 is moderate (particularly for small models) since finding at least one parameter point at which the rank condition holds may be accomplished in an ad–hoc way (as in RWZ, p. 679). However when there are many non–zero restrictions and the model is not small practicability of such an ad–hoc method becomes limited. It is useful to have a mechanical method to do that.

[2] The approach is quite similar to that exploited in Johansen (1995).

From now on (but without loss of generality) we confine our reasoning to arbitrary $j \in \{1,\ldots,n-1\}$. Since all restrictions are homogenous we can always find $H_j : k \times (k-q_j)$ with $rank(H_j) = k - q_j$ such that $f_j = H_j g_j$ (columns of H_j form a basis for the null space of Q_j so that $Q_j H_j = 0$) and $g_j : (k-q_j) \times 1$ are free elements in f_j. Using this notation our necessary and sufficient condition for regional identification (confined to a particular j) reads

$$rank(Q_j H_{j+1} g_{j+1} \vdots Q_j H_{j+2} g_{j+2} \vdots \ldots \vdots Q_j H_n g_n) = n - j \qquad (5)$$

Define $\mathcal{A}_i = Q_j H_{j+i}$ and denote $\mathcal{A} = \{\mathcal{A}_i : i = 1,\ldots,n-j\}$. Note that $\mathcal{A}$ is a finite collection of (not necessarily distinct) finite subsets of the vector space of dimension q_j. Think of $\mathcal{A}_i$ as the collection of its column vectors. It turns out that the question whether there is at least one parameter point such that $rank(Q_j f_{[j+1]}) = rank(Q_j H_{j+1} g_{j+1} \vdots Q_j H_{j+2} g_{j+2} \vdots \ldots \vdots Q_j H_n g_n) = n - j$ is equivalent to the question about the existence of the independent transversal in $\mathcal{A}$.

Definition 3: *The independent transversal is a set $T = \{a_i : i = 1,\ldots,n-j\}$ such that $a_i \in \mathcal{A}_i$, $a_i \neq a_j$ for $i \neq j$ and the collection $\{a_1, a_2, \ldots, a_{n-j}\}$ is linearly independent.*

Intuitively, if there is an independent transversal in $\mathcal{A}$ then one may take all g_{j+i}, for $i = 1,\ldots,n-j$, to be vectors with one element equal to 1 and all the remaining equal to 0's. In such a case the role of each g_{j+i} is to select some element a_i of the independent transversal from $\mathcal{A}_i = Q_j H_{j+i}$, which is one of the columns of $\mathcal{A}_i$.

Theorem 1: *A collection $\mathcal{A}$ possesses an independent transversal ifif for all $r = 1,\ldots,n-j$ and all sets of indices $1 \leq i_1 < i_2 < \ldots < i_r \leq n-j$, $rank(Q_j H_{j+i_1} \vdots Q_j H_{j+i_2} \vdots \ldots \vdots Q_j H_{j+i_r}) \geq r$.*

Proof: This is a version of Rado's theorem, see e.g. Mirsky (1971), ch. 6. In particular it is based on observation that $\dim(sp\{\bigcup_{1\leq i \leq k} \mathcal{A}_i\}) \equiv \dim(sp\{\mathcal{A}_1, \ldots, \mathcal{A}_k\}) = rank(\mathcal{A}_1 \vdots \ldots \vdots \mathcal{A}_k)$ for any k, where $sp\{X\}$ is the subspace spanned by X and $\dim(\cdot)$ is its dimension.

If $\mathcal{A}$ has an independent transversal then one may choose g_{j+i}'s such that $rank(Q_j f_{[j+1]}) = rank(Q_j H_{j+1} g_{j+1} \vdots Q_j H_{j+2} g_{j+2} \vdots \ldots \vdots Q_j H_n g_n) = n - j$. In light of theorem 3 in RWZ we have a necessary and sufficient condition for $rank(Q_j f_{[j+1]}) = n - j$, almost everywhere

Proposition 4: *Let* $q_1 \geq q_2 \geq \ldots \geq q_n$. *Then for any fixed* $j \in \{1,\ldots,n-1\}$, $rank(Q_j f_{[j+1]}) = n - j$, *almost everywhere ifif for all* $r = 1,\ldots,n-j$ *and all sets of indices* $1 \leq i_1 < i_2 < \ldots < i_r \leq n - j$, $rank(Q_j H_{j+i_1} \vdots Q_j H_{j+i_2} \vdots \ldots \vdots Q_j H_{j+i_r}) \geq r$.

Needless to say, to check if the SVAR is regionally identified almost everywhere we have to apply proposition 4 for all $j \in \{1,\ldots,n-1\}$. Although this may be cumbersome for large models it should be emphasized that the criterion is operational since all elements in Q_j and H_j are known a priori.

V. EXAMPLES

To fully appreciate our results concerning the rank condition (proposition 1) and the order condition (corollary 1) consider the identifying scheme carefully derived by Sims and Zha (2006) to approximate the Dynamic Stochastic General Equilibrium model, labeled therein as M2 model[3].

$$A_0 = \begin{array}{r} \\ Pcm \\ M \\ R \\ Pim \\ Py \\ W \\ y \\ Tbk \end{array} \begin{array}{c} \begin{array}{cccccccc} Tbk & MD & y & W & MS & Py & Pim & Pcm \end{array} \\ \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ 0 & a_{22} & 0 & 0 & a_{25} & 0 & 0 & a_{28} \\ 0 & a_{32} & 0 & 0 & a_{35} & 0 & 0 & a_{38} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{47} & a_{48} \\ 0 & -a_{22} & 0 & 0 & 0 & a_{56} & a_{57} & a_{58} \\ 0 & 0 & 0 & a_{64} & 0 & a_{66} & a_{67} & a_{68} \\ 0 & -a_{22} & a_{73} & a_{74} & 0 & a_{76} & a_{77} & a_{78} \\ a_{81} & 0 & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{bmatrix} \end{array} \qquad (6)$$

To be sure, (6) differs from that used in Sims and Zha (2006) only in the irrelevant aspect i.e. we permuted equations so as $q_1 \geq q_2 \geq \ldots \geq q_8$ and in our convention the columns in A_0 correspond to model's equations. Each column in (6) represents the behavioral equation labeled at the top. Among all equations only the second one (labeled *MD*, which stands for money demand) and the fifth one (labeled *MS*, which

[3] In fact all conclusions to be made are the same for the other model estimated in Sims and Zha (2006) i.e. TR model. We note in passing that the SVAR under identifying scheme (6) appeared also as a leading example of non–recursive SVAR in the influential survey by Christiano et al. (1999). This strengthens the importance of our verdict concerning the identifiability of SVAR under (6).

stands for money supply) have serious economic interpretation. The remaining ones are labeled by variables appearing in the model: Pcm: producers' price index for intermediate goods, M: M2, R: federal funds rate, Pim: producers' price index for intermediate materials, Py: GNP deflator, W: average hourly earnings of non–agricultural workers, y: real GNP, Tbk: bankruptcy filings (personal and business).

This SVAR is over–identified in the sense that there are in total 30 restrictions imposed on A_0 whereas the common order condition for exact identification is $\frac{1}{2}n(n-1)=28$. Using theory from RWZ the only way to find out whether the model is identified or not is to construct $rank(M_j(f(A_0,A_+)))$ for each $j=1,\ldots,n$. If we will manage to guess one artificial point at which $rank(M_j(f(A_0,A_+)))=n$ for each $j=1,\ldots,n$, then the SVAR is identified almost everywhere. But the relevant question is this. What if we can not arrive at this point or even if we demonstrate that finding such a point is impossible. Since RWZ "qualified" their condition as only sufficient it is logically possible that the SVAR is still identified almost everywhere. In this respect our contribution is welcomed since we showed that if we do demonstrate that finding such a point is impossible then the SVAR is definitely non–identified. On the other hand, the usefulness of our order condition applied to (6) is invaluable. The reason is that we do not have to construct "big objects" $rank(M_j(f(A_0,A_+)))$ for each $j=1,\ldots,n$, at all, to decide whether identifying scheme (6) leads to identified SVAR. It is easy to observe that the model violates the order condition since the first equation consists of only 6 restrictions whereas the order condition requires that we should impose at least 7 restrictions on the first equation. We conclude that the SVAR models M2 and TR adopted by Sims and Zha (2006) are not identified.

Next we provide two examples to illustrate results from section IV. The first one deals only with exclusive restrictions and is taken from RWZ, section 5.2. Consider the monetary SVAR in which all "zero" restrictions are imposed on a matrix of contemporaneous relations

$$A_0 = \begin{matrix} \\ \log Y \\ \log P \\ R \\ \log M \\ \log P_c \end{matrix} \begin{matrix} \begin{matrix} PS & PS & MP & MD & Inf \end{matrix} \\ \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & a_{15} \\ 0 & a_{22} & 0 & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix} \end{matrix} \tag{7}$$

The columns in (7) stand for equations which are (roughly) described by its economic interpretation: *PS* – production sector, *MP* – monetary policy, *MD* – money demand and *Inf* – information. The variables are log GDP $(\log Y)$, log GDP deflator $(\log P)$, the nominal interest rate (R), log M3 $(\log M)$ and log commodity prices $(\log P_c)$. To establish identification almost everywhere we use our proposition 3. To this end

$$Q_1 = \begin{bmatrix} 0\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0 \\ 0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,1 \end{bmatrix},\ Q_2 = \begin{bmatrix} 0\,0\,1\,0\,0 \\ 0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,1 \end{bmatrix},\ Q_3 = \begin{bmatrix} 1\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1 \end{bmatrix},\ Q_4 = \begin{bmatrix} 0\,0\,0\,0\,1 \end{bmatrix}$$

hence $q_1 \geq q_2 \geq \ldots \geq q_5$, as required. Moreover since in this case $f(A_0, A_+) = A_0$, f_j is just the $j-$th column of A_0. Hence

$$Q_1 f_{[2]} \equiv Q_1[f_2 f_3 f_4 f_5] = \begin{bmatrix} a_{22} & 0 & a_{24} & a_{25} \\ 0 & a_{33} & a_{34} & a_{35} \\ 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{55} \end{bmatrix}, \qquad Q_2 f_{[3]} \equiv Q_2[f_3 f_4 f_5] = \begin{bmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ 0 & 0 & a_{55} \end{bmatrix},$$

$$Q_3 f_{[4]} \equiv Q_3[f_4 f_5] = \begin{bmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \\ 0 & a_{55} \end{bmatrix}, \qquad Q_4 f_{[5]} \equiv Q_4 f_5 = a_{55}$$

Evidently $\rho(Q_j f_{[j+1]}) = 5 - j$, for all $j = 1,2,3,4$ (e.g. take $\{a_{55}, a_{24}, a_{34}, a_{43}, a_{22}\}$), thus the SVAR with identifying scheme (7) is regionally identified almost everywhere.

To illustrate the utilization of proposition 4 consider the following identifying scheme imposed on a matrix of simultaneous relations

$$A_0 = \begin{array}{r} \\ R \\ \log M \\ \log P \\ \log Y \end{array} \begin{array}{c} \begin{array}{cccc} PS & MD & PS & MP \end{array} \\ \begin{bmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & -a_{22} & a_{33} & 0 \\ a_{41} & -a_{22} & a_{43} & 0 \end{bmatrix} \end{array} \tag{8}$$

where the symbols for the variables and equations' labels are precisely the same as in (7). Note that $q_1 \geq q_2 \geq q_3 \geq q_4$ and $f(A_0, A_+) = A_0$. Since the *MD* equation is identified only by "non–zero" restrictions we can not use proposition 3. Instead we may apply proposition 4. To this end

$$Q_1 = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix}; H_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; Q_2 = \begin{bmatrix} 0\,1\,1\,0 \\ 0\,1\,0\,1 \end{bmatrix}; H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}; Q_3 = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{bmatrix}; H_3 = \begin{bmatrix} 0\,0 \\ 0\,0 \\ 1\,0 \\ 0\,1 \end{bmatrix}; H_4 = \begin{bmatrix} 1\,0 \\ 0\,1 \\ 0\,0 \\ 0\,0 \end{bmatrix}$$

a) Checking $rank(Q_1 f_{[2]}) = n - 1 = 3$:

$$rank(Q_1 H_2) = rank\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \geq 1, rank(Q_1 H_3) = rank\begin{pmatrix} 0\,0 \\ 0\,0 \\ 1\,0 \end{pmatrix} \geq 1, rank(Q_1 H_4) = rank\begin{pmatrix} 1\,0 \\ 0\,1 \\ 0\,0 \end{pmatrix} \geq 1$$

$$rank(Q_1(H_2 \vdots H_3)) = rank\begin{pmatrix} 1 & 0 & 0\,0 \\ 0 & 1 & 0\,0 \\ 0 & -1 & 1\,0 \end{pmatrix} \geq 2,\ rank(Q_1(H_2 \vdots H_4)) = rank\begin{pmatrix} 1 & 0 & 1\,0 \\ 0 & 1 & 0\,1 \\ 0 & -1 & 0\,0 \end{pmatrix} \geq 2,$$

$$rank(Q_1(H_3 \vdots H_4)) = rank\begin{pmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 1\,0\,0\,0 \end{pmatrix} \geq 2$$

$$rank(Q_1(H_2 \vdots H_3 \vdots H_4)) = rank\begin{pmatrix} 1 & 0 & 0\,0\,1\,0 \\ 0 & 1 & 0\,0\,0\,1 \\ 0 & -1 & 1\,0\,0\,0 \end{pmatrix} \geq 3$$

b) Checking $rank(Q_2 f_{[3]}) = n - 2 = 2$:

$$rank(Q_2 H_3) = rank\begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} \geq 1,\ rank(Q_2 H_4) = rank\begin{pmatrix} 0\,1 \\ 0\,1 \end{pmatrix} \geq 1$$

$$rank(Q_2(H_3 \vdots H_4)) = rank\begin{pmatrix} 1\,0\,0\,1 \\ 0\,1\,0\,1 \end{pmatrix} \geq 2$$

c) Checking $rank(Q_3 f_{[4]}) = n - 3 = 1$:

$$rank(Q_3 H_4) = rank\begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} \geq 1$$

Since the necessary and sufficient condition of proposition 4 holds, the SVAR with identifying scheme (8) is regionally identified almost everywhere.

VI. CONCLUSIONS

Since SVAR model is still the empirical window though which we observe macroeconomic world, the fundamental question about identification of SVAR models is very important. Hence the path–breaking contribution of Rubio–Ramírez et al (2010) should be fully acknowledged. The present paper provided some further results concerning this aspect of the SVAR methodology, which in our opinion are worth knowing.

APPENDIX 1:

Our goal is to show that $rank(Q_j f_{[j+1]}) = n - j$, for $j = 1,\ldots,n-1$ if and only if $Q_j f(A_0 P, A_+ P)e_j = 0$; for $j = 1,\ldots,n$, implies $P \in D$. We should proceed sequentially beginning with $j = 1$. By assumption we have $Q_1 f(A_0, A_+)e_1 = 0$. Note that $Q_1 f(A_0 P, A_+ P)e_1 = Q_1 f(A_0, A_+)Pe_1 = Q_1 f(A_0, A_+)p_{\bullet 1}$, where $p_{\bullet i}$ is the $i-$th column of P. Let us introduce the notation $p_{[j]1} = (p_{j1},\ldots,p_{n1})'$. It follows $p_{\bullet 1} = (p_{11}, p_{21},\ldots,p_{n1})' = (p_{11}, p_{[2]1}')'$. Then $0 = Q_1 f(A_0, A_+)p_{\bullet 1} = Q_1(f_1 p_{11} + f_{[2]}p_{[2]1}) =$ $= Q_1 f_{[2]} p_{[2]1}$ (note $Q_1 f_1 = 0$ since $Q_1 f_1 \equiv Q_1 f(A_0, A_+)e_1$). For regional identification we should have $p_{[2]1} = 0$. But $Q_1 f_{[2]} p_{[2]1} = 0 \Rightarrow p_{[2]1} = 0$ if and only if $Q_1 f_{[2]}$ has full column rank i.e. $rank(Q_1 f_{[2]}) = n - 1$. In this case we may say that the first equation is regionally identified (provided that we permuted SVAR equations so as $q_1 \geq q_2 \geq \ldots \geq q_n$). Now $p_{[2]1} = 0$ implies by the orthogonal restriction that $p_{11} = \pm 1$ and $p_{12} = 0$. Thus for $j = 2$ we get $Q_2 f(A_0 P, A_+ P)e_2 = Q_2 f(A_0, A_+)p_{\bullet 2} =$ $= Q_2(f_1 p_{12} + f_2 p_{22} + f_{[3]}p_{[3]2}) = 0$, where $p_{[j]2} = (p_{j2},\ldots,p_{n2})'$. Since $p_{12} = 0$ and $Q_2 f_2 = 0$ we have $Q_2 f(A_0 P, A_+ P)e_2 = Q_2 f_{[3]} p_{[3]2} = 0$. But $Q_2 f_{[3]} p_{[3]2} = 0 \Rightarrow p_{[3]2} = 0$ if and only if $rank(Q_2 f_{[3]}) = n - 2$. If the latter condition holds then by the orthogonality we have $p_{22} = \pm 1$. For $j = 3$, $Q_3 f(A_0 P, A_+ P)e_3 = Q_3(f_1 p_{13} + f_2 p_{23} + f_3 p_{33} + f_{[4]}p_{[4]3}) = 0$. Since $p_{\bullet 3}$ must be orthogonal to $p_{\bullet 1}$ and $p_{\bullet 2}$ this implies $p_{13} = 0$, $p_{23} = 0$. Since also $Q_3 f_3 = 0$, we have $Q_3 f(A_0 P, A_+ P)e_3 = Q_3 f_{[4]} p_{[4]3} = 0$. The latter implies $p_{[4]3} = 0$ if and only if $rank(Q_3 f_{[4]}) = n - 3$. Of course if $p_{[4]3} = 0$ then by the orthogonality $p_{33} = \pm 1$. The rest of proof follows sequentially but ends with $j = n - 1$ since if the first $n - 1$ columns of P are demonstrated to be the first $n - 1$ columns of the diagonal matrix with ± 1 on the diagonal then, by the orthogonality, the last column of P must be $(00\ldots 0 \pm 1)'$.

APPENDIX 2:

Let us denote the generic elements of $Q_j f_{[j+1]}$ as $x_{l,k}$ for $l = 1,\ldots,q_j$; $k = 1,\ldots,n-j$. Suppose $rank(Q_j f_{[j+1]}) = n - j$, for all $j = 1,\ldots,n-1$, almost everywhere. Then $rank(Q_j f_{[j+1]}) = n - j$ for arbitrary $j \in \{1,\ldots,n-1\}$ at some (A_0, A_+). It follows that there exists a submatrix of $Q_j f_{[j+1]}$ of dimension $(n-j)\times(n-j)$, say K_j, such that $\det(K_j) \neq 0$. Since the determinant is an alternating sum of all permutation products (see e.g. Mirsky (1955), ch. 1, for the precise meaning of this), it follows that at least one permutation product is non–zero. But this just means $\rho(Q_j f_{[j+1]}) = n - j$ for arbitrary $j \in \{1,\ldots,n-1\}$.

Now let $\rho(Q_j f_{[j+1]}) = n - j$ for arbitrary $j \in \{1,\ldots,n-1\}$. Then there is at least one permutation product in $Q_j f_{[j+1]}$, say $x_\pi \equiv x_{i_1,1} x_{i_2,2} \ldots x_{i_{n-j},n-j}$, where $(i_1, i_2, \ldots, i_{n-j})$ is an $(n-j)-$permutation of the integers $1, 2, \ldots, q_j$, which is not identically equal to zero. Let us set all elements of $Q_j f_{[j+1]}$ except those comprising the given x_π to zero (this is possible since we have only exclusive restrictions). In other words in the first column of $Q_j f_{[j+1]}$ all elements except $x_{i_1,1}$ are set to zero, in the second one all elements except $x_{i_2,2}$ are set to zero, etc. Note that such a point belongs to the restricted parameter space. Whenever $x_{i_1,1} \neq 0, x_{i_2,2} \neq 0, \ldots, x_{i_{n-j},n-j} \neq 0$ it follows that at such a constructed parameter point $rank(Q_j f_{[j+1]}) = n - j$ for arbitrary $j \in \{1,\ldots,n-1\}$, hence for all j. By proposition 2 this is equivalent to $rank(M_j(f(A_0, A_+))) = n$ for all $j = 1,\ldots,n$ at such a constructed parameter point. By theorem 3 in RWZ, $rank(M_j(f(A_0, A_+))) = n$ for all $j = 1,\ldots,n$ almost everywhere hence $rank(Q_j f_{[j+1]}) = n - j$, for all $j = 1,\ldots,n-1$, almost everywhere.

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