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Anticipated Alternative Instrument-Rate Paths in Policy Simulations

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Abstract

This paper specifies how to do policy simulations with alternative instrument-rate paths in DSGE models such as Ramses, the Riksbank's main model for policy analysis and forecasting. The new element is that these alternative instrument-rate paths are *anticipated* by the private sector. Such simulations correspond to situations where the Riksbank transparently announces that it plans to implement a particular instrument-rate path and where this announcement is believed by the private sector. Previous methods have instead implemented alternative instrument-rate paths by adding unanticipated shocks to an instrument rule, as in the method of modest interventions by Leeper and Zha (2003). This corresponds to a very different situation where the Riksbank would nontransparently and secretly plan to implement deviations from an announced instrument rule. Such deviations are in practical simulations normally both serially correlated and large, which seems inconsistent with the assumption that they would remain unanticipated by the private sector. Simulations with anticipated instrument-rate paths seem more relevant for the transparent flexible inflation targeting that the Riksbank conducts. We provide an algorithm for the computation of policy simulations with arbitrary restrictions on nominal and real instrument-rate paths for an arbitrary number of periods after which a given policy rule, including targeting rules and explicit, implicit, or forecast-based instrument rules is implemented. When inflation projections are sufficiently sensitive to the real interest-rate path, restrictions on real interest-rate paths provide more intuitive and robust results, whereas restrictions on nominal interest-rate path may provide somewhat counter-intuitive results.

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1. Introduction

This paper specifies how to do policy simulations with alternative instrument-rate paths in DSGE models, such as Ramses, the Riksbank's main dynamic stochastic general equilibrium (DSGE) model.[1] The new element is that these alternative instrument-rate paths are *anticipated* by the private sector. Such simulations correspond to situations where the central bank transparently announces that it plans to implement a particular instrument-rate path and where this announced plan for the instrument rate is believed by the private sector. Previous methods have instead implemented alternative instrument-rate paths by adding unanticipated shocks to an instrument rule, as in the method of modest interventions by Leeper and Zha [14] (see appendix A). That method is designed to deal with policy simulations that involve "modest" unanticipated deviations from a policy rule. It corresponds to a very different situation when the central bank would nontransparently and secretly plan to surprise the private sector by deviations from an announced instrument rule. Aside from corresponding to very non-transparent policy, such deviations are in practical simulations usually both serially correlated and large, which seems inconsistent with the assumption that they would remain unanticipated by the private sector. In other words, they are in practice not "modest" in the sense of Leeper and Zha. Simulations with anticipated instrument-rate paths seem more relevant for the transparent flexible inflation targeting that central banks such as the Riksbank conduct.

The main purpose with policy simulations with alternative instrument-rate paths is to provide the policymaker with a set of policy choices and to illustrate how the development of the economy would differ for different policy choices. Doing alternative policy simulations by optimal policy simulations for alternative weights on target variables in the loss function has the advantage that the different alternatives are efficient, for instance, in the sense that further stabilization of the inflation projection around the inflation target cannot be achieved without allowing more fluctuations in the projection of the output gap.

In this paper we instead show how to do alternative policy simulations in a different way, namely by satisfying arbitrary restrictions on the nominal or real instrument rate by adding an anticipated sequence of constants to a general but constant policy rule, including targeting rules (conditions on target variables) and explicit or implicit instrument rules (instrument rules where the instrument rate depends on predetermined variables only or also forward-looking variables). By applying a

[1] The instrument rate is the short interest rate that the central bank uses as a (policy) instrument (control variable). For the Riksbank, the instrument rate is the repo rate.

special case of the more general analysis of judgment in Svensson [23], this can be done in a very simple way by treating the vector of such constants as an additional predetermined variable. This allows the model to be written in the standard state-space form and the use of standard solution methods. This very conveniently allows policy simulations for alternative arbitrary nominal and real instrument-rate paths, whether or not these are optimal for a particular reasonable loss function or not. We also provide, in an appendix, an alternative somewhat more complicated algorithm for arbitrary time-varying policy rules.[2]

We consider policy simulations where restrictions on the nominal or real instrument-rate path are eventually followed by an anticipated future switch to a given well-behaved policy rule, either optimal or arbitrary. With such a setup, there is a unique equilibrium for each specified set of restrictions on the nominal or real instrument-rate path. The equilibrium will, in a model with forward-looking variables, depend on which future policy rule is implemented, but for any given such policy rule, the equilibrium is unique. It is well known since Sargent and Wallace [19] that an exogenous nominal instrument-rate path will normally lead to indeterminacy in a model with forward-looking variables (and to an explosive development in a model with backward-looking model), so at some future time the nominal instrument-rate must become endogenous for a well-behaved equilibrium to result (see also Gagnon and Henderson [11]). Such a setup with a switch to a well-behaved policy rule solves the problem with multiple equilibria for alternative instrument-rate projections that Gali [12] has emphasized. On the other hand, consistent with Gali's results, the unique equilibrium depends on and is sensitive to both the time of the switch and the policy rule to which policy shifts.

In describing the optimal policy choice, we find it practical to describe it as the selection of an optimal policy projection of the instrument rate and the target variables from the set of feasible policy projections, the set of projections of the instrument rate and the target variables that satisfy the projection model of the economy. Since the parameters of the projection model are interpreted as structural parameters and the projection of the instrument-rate path is anticipated by the private sector, the description is not subject to the Lucas [16] critique. Thus, the optimal policy projection is the feasible projection that minimizes the intertemporal loss function that represents the objectives of monetary policy. Although this policy choice is formally equivalent to the choice of

[2] Linear rational-expectations models with anticipated arbitrary instrument-rate paths implemented via anticipated time-varying intercepts gives rise to systems of linear difference equations that are analogous to those with stochastic disturbances with arbitrary time-varying nonzero means. As explained in Svensson [25], these can also be solved with several alternative algorithms, including the AIM algorithm of Anderson and Moore [7]-[10], the Gensys algorithm of Sims [20], and the generalizatio of the Klein [13] algorithm of Svensson [23]. The algorithm used here is arguably especially convenient, though.

an optimal instrument rule, we find the above description of the optimal policy choice closer to the actual choice situation of a monetary-policy committee such as the Riksbank's Executive Board. Riksbank Board members see graphs of alternative policy projections to choose from. They don't see a set of alternative instrument rules to choose from. In particular, they don't see an explicit representation of an optimal instrument rule. Indeed, an optimal instrument rule implicitly takes into account all information that has an impact on the forecast of the target variables, including judgment, which makes it more or less infeasible to write it down.

The set of feasible policy projections has an infinite number of elements. The methods to construct alternative policy projections discussed in this paper can be seen as restricting the policy choice to a finite number of relevant alternative policy projections.

We show our results for three different models, namely the small empirical backward-looking model of the U.S. economy of Rudebusch and Svensson [18], the small empirical forward-looking model of the U.S. economy of Lindé [15], and Ramses, the medium-sized model of the Swedish economy of Adolfson, Laséen, Lindé, and Villani [4].[3] From the analysis in this paper, we conclude that, in a model without forward-looking variables, such as the Rudebusch-Svensson model, there is no difference between policy simulations with anticipated and unanticipated restrictions on the instrument-rate path. In a model with forward-looking variables, such as the Lindé model or Ramses, there is such a difference, and the impact of anticipated restrictions would generally be larger than that of unanticipated restrictions. In a model with forward-looking variables, exogenous restrictions on the instrument-rate path are consistent with a unique equilibrium, if there is a switch a well-behaved policy rule in the future. For given restrictions on the instrument-rate path, the equilibrium depends on that policy rule.

Furthermore, our analysis shows that, if inflation is sufficiently sensitive to the real instrument rate, "unusual" equilibria may result from restrictions on the nominal instrument rate. Such cases have the property that a shift up of the real interest-rate path reduces inflation and inflation-expectations so much that the nominal interest-rate path (which by the Fisher equation equals the real interest-rate path plus the path of inflation expectations) shifts down. Then, a shift up of the nominal interest-rate path requires an equilibrium where the path of inflation and inflation expectations shifts up more and the real instrument-rate path shifts down. In the Rudebusch-Svensson model, which has no forward-looking variables, inflation is so sluggish and insensitive to changes in the real instrument rate that there are only small differences between restrictions

[3] See also Adolfson, Laséen, Lindé, and Villani [5] and [6].

on the nominal and real instrument rate. In the Lindé model, inflation is so sensitive to the real instrument rate that restrictions for 5–6 quarters or more on the nominal instrument rate results in unusual equilibria. In Ramses, unusual equilibrium seem to require restrictions for 11 quarters or more.

Because of the possibility of unusual equilibria for restrictions on nominal instrument rates, restrictions on the real rather than nominal instrument rate are preferable since they lead to more robust and intuitive results. Also, it makes sense to impose restrictions on and consider shifts of the real instrument-rate path rather than the nominal one, since the real interest rates rather than nominal ones are what matter for real activity and for inflation. In general, given the possibility of unusual equilibria, it may in many cases be preferable to instead generate alternative instrument-rate projections as optimal policy projections that minimize an intertemporal loss function for alternative weights on the target variables. Adolfson, Laséen, Lindé, and Svensson [1] (ALLS1) show how to do optimal policy simulations in Ramses, that is, simulations of policy that minimizes an intertemporal loss function under commitment.[4]

The paper is organized as follows: Section 2 presents the state-space representation of a linear(ized) DSGE model, section 3 describes the projection model. Section 4 describes the optimal policy choice, and in particular emphasizes that the real-world policy problem for monetary-policy makers can be seen as selecting an optimal projection in a set of feasible projections, rather than as selecting a particular instrument rule (although it will be equivalent to choosing an optimal instrument rule, but that instrument rule is in practice too complex to be made explicit). Section 5 specifies optimal projections. Some of the material of sections 2 and 5 is discussed in more detail in ALLS1. Section 6 shows how to do policy simulations with an arbitrary constant (that is, time-invariant) policy rule, such as an instrument rule or a targeting rule. Section 7 shows a very simple way of doing policy simulations that satisfy arbitrary time-varying restrictions on the real or nominal instrument rate by introducing anticipated time-varying constants in the policy rule. These sections also highlight the difference between models with forward-looking variables such as Ramses and the Lindé models and previous models without forward-looking variables such as the Rudebusch-Svensson model. Section 8 provides examples of restrictions on nominal and real instrument-rate paths for the Rudebusch-Svensson model, the Lindé model, and Ramses. These

[4] The intertemporal loss function is a discounted sum of expected future period losses. The period loss is a quadratic loss function that corresponds to flexible inflation targeting and is the weighted sum of three terms: the squared inflation gap between inflation and the inflation target, the squared output gap between output and potential output, and the squared quarterly change in the Riksbank's instrument rate, the repo rate. Alternative optimal policy simulations can be done by changing the weights and arguments in the loss function.

examples also highlight the difference between restrictions on the nominal and the real instrument-rate paths. These differences are important when inflation projections and inflation expectations are sensitive to changes in the real instrument-rate paths. Section 9 presents some conclusions.

Appendix A demonstrates the Leeper and Zha [14] method of modest interventions in this framework. Appendix B demonstrates an alternative method of doing policy simulations with an arbitrary time-varying policy rule that can incorporate time-varying restrictions on the real and nominal instrument rate. Appendices C and D provide some details on the Rudebusch-Svensson and Lindé models, respectively.

2. The model

A linear model with forward-looking variables can be written in the following practical state-space form,

$$\begin{bmatrix} X_{t+1} \\ Hx_{t+1|t} \end{bmatrix} = A \begin{bmatrix} X_t \\ x_t \end{bmatrix} + Bi_t + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1}. \tag{2.1}$$

Here, X_t is an n_X-vector of *predetermined* variables in period t (where the period is a quarter); x_t is an n_x-vector of *forward-looking* variables; i_t is generally an n_i-vector of (policy) *instruments* but in the cases examined here it is a scalar, the instrument rate, in the Riksbank's case the repo rate, so $n_i = 1$; ε_t is an n_ε-vector of i.i.d. shocks with mean zero and covariance matrix I_{n_ε}; A, B, and C, and H are matrices of the appropriate dimension; and $y_{t+\tau|t}$ denotes $\mathrm{E}_t y_{t+\tau}$ for any variable y_t, the rational expectation of $y_{t+\tau}$ conditional on information available in period t. The forward-looking variables and the instruments are the *nonpredetermined* variables.[5]

The variables can be measured as differences from steady-state values, in which case their unconditional means are zero. Alternatively, one of the components of X_t can be unity, so as to allow the variables to have nonzero means. The elements of the matrices A, B, C, and H are estimated with Bayesian methods and considered fixed and known for the policy simulations. Hence the conditions for certainty equivalence are satisfied.

The upper block of (2.1) provides n_X equations determining the n_X-vector X_{t+1} in period $t+1$ for given X_t, x_t, i_t and ε_{t+1},

$$X_{t+1} = A_{11}X_t + A_{12}x_t + B_1 i_t + C\varepsilon_{t+1}, \tag{2.2}$$

[5] A variable is predetermined if its one-period-ahead prediction error is an exogenous stochastic process (Klein [13]). For (2.1), the one-period-ahead prediction error of the predetermined variables is the stochastic vector $C\varepsilon_{t+1}$.

where A and B are partitioned conformably with X_t and x_t as

$$A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{2.3}$$

The lower block provides n_x equations determining x_t in period t for given $x_{t+1|t}$, X_t, and i_t,

$$x_t = A_{22}^{-1}(Hx_{t+1|t} - A_{21}X_t - B_2 i_t). \tag{2.4}$$

We hence assume that the $n_x \times n_x$ submatrix A_{22} is nonsingular.[6]

Let Y_t be an n_Y-vector of *target* variables, measured as the difference from an n_Y-vector Y^* of *target levels*. This is not restrictive, as long as we keep the target levels time invariant. If we would like to examine the consequences of different target levels, we can instead interpret Y_t as the absolute level of the target levels and replace Y_t by $Y_t - Y^*$ everywhere below. We assume that the target variables can be written as a linear function of the predetermined, forward-looking, and instrument variables,

$$Y_t = D \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} \equiv [D_X \quad D_x \quad D_i] \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}, \tag{2.5}$$

where D is an $n_Y \times (n_X + n_x + n_i)$ matrix and partitioned conformably with X_t, x_t, and i_t. For plotting and other purposes, and to avoid unnecessary separate program code, it is convenient to expand the vector Y_t to include a number of variables of interest that are not necessary target variables or potential target variables. These will then have zero weight in the loss function.

Let the intertemporal loss function in period t be

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau}, \tag{2.6}$$

where $0 < \delta < 1$ is a discount factor, L_t is the period loss given by

$$L_t \equiv Y_t' W Y_t, \tag{2.7}$$

and W is a symmetric positive semidefinite matrix containing the weights on the individual target variables.

In a backward-looking model, that is, a model without forward-looking variables, there is no vector x_t of forward-looking variables, no lower block of equations in (2.1), and the vector of target variables Y_t only depends on the vector of predetermined variables X_t and the (vector of) instrument(s) i_t.

[6] Without loss of generality, we assume that the shocks ε_t only enter in the upper block of (2.1), since any shocks in the lower block of (2.1) can be redifined as additional predetermined variables and introduced in the upper block.

3. The projection model and the feasible set of projections

Let $u^t \equiv \{u_{t+\tau,t}\}_{\tau=0}^{\infty}$ denote a *projection* in period t for any vector of variables u_t, where $u_{t+\tau,t}$ denotes the mean forecast of the realization of the vector in period $t+\tau$ conditional on information available in period t. We refer to τ as the horizon of the forecast $u_{t+\tau,t}$. The *projection model* for the projections (X^t, x^t, i^t, Y^t) in period t uses that the projection of the zero-mean i.i.d. shocks is zero, $\varepsilon_{t+\tau,t} = 0$ for $\tau \geq 1$. It can then be written as

$$\begin{bmatrix} X_{t+\tau+1,t} \\ Hx_{t+\tau+1,t} \end{bmatrix} = A \begin{bmatrix} X_{t+\tau,t} \\ x_{t+\tau,t} \end{bmatrix} + Bi_{t+\tau,t}, \tag{3.1}$$

$$Y_{t+\tau,t} = D \begin{bmatrix} X_{t+\tau,t} \\ x_{t+\tau,t} \\ i_{t+\tau,t} \end{bmatrix}, \tag{3.2}$$

for $\tau \geq 0$, where

$$X_{t,t} = X_{t|t}, \tag{3.3}$$

where $X_{t|t}$ is the estimate of predetermined variables in period t conditional on information available in the beginning of period t. Thus, "$,t$" and "$|t$" in subindices refer to projections (forecasting) and estimates ("nowcasting" and "backcasting") in the beginning of period t, respectively. The *feasible set of projections* for given $X_{t|t}$, denoted $\mathcal{T}(X_{t|t})$, is the set of projections that satisfy (3.1)-(3.3). We call $\mathcal{T}_t(X_{t|t})$ the set of feasible projections in period t. It is conditional on the estimates of the matrices A, B, and H and the estimate of the current realization of the predetermined variables $X_{t|t}$.

In a backward-looking model, there are no forward-looking variables in the projection model.

4. Optimal policy choice

The policy problem in period t is to determine the optimal projection in period t. The optimal projection is the projection $(\hat{X}^t, \hat{x}^t, \hat{\imath}^t, \hat{Y}^t)$ that minimizes the intertemporal loss function,

$$\sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau,t}, \tag{4.1}$$

where the period loss, $L_{t+\tau,t}$, is specified as

$$L_{t+\tau,t} = Y_{t+\tau,t}{}' W Y_{t+\tau,t} \tag{4.2}$$

for $\tau \geq 0$. The minimization is subject to the projection being in the feasible set of projections for given $X_{t|t}$, $\mathcal{T}(X_{t|t})$.

When the policy problem is formulated in terms of projections, we can allow $0 < \delta \leq 1$, since the above infinite sum will normally converge also for $\delta = 1$. The optimization is done under commitment in a timeless perspective (Woodford [27]).

The intertemporal loss function (4.1) with the period loss function (4.2) introduces a preference ordering over projections of the target variables, Y^t. We can express this preference ordering as the modified intertemporal loss function,

$$\mathcal{L}_t(Y^t) + \frac{1}{\delta}\Xi'_{t-1}(x_{t,t} - x_{t,t-1}) \equiv \sum_{\tau=0}^{\infty} \delta^\tau Y'_{t+\tau,t} W Y_{t+\tau,t} + \frac{1}{\delta}\Xi'_{t-1}(x_{t,t} - x_{t,t-1}), \tag{4.3}$$

where the modification is the added term $\frac{1}{\delta}\Xi'_{t-1}(x_{t,t} - x_{t,t-1})$. In that term, Ξ_{t-1} is the vector of Lagrange multipliers for the equations for the forward-looking variables from the optimization problem in period $t-1$, $x_{t,t}$ is the projection of the vector of forward-looking variables in period t that satisfies the projection model (3.1) and the initial condition (3.3), and $x_{t,t-1}$ is the optimal projection in period $t-1$ of the vector of forward-looking variables in period t ($x_{t,t-1}$ is predetermined in period t and normalizes the added term and makes it zero in case the projection $x_{t,t}$ coincides with the projection $x_{t,t-1}$ but does not affect the choice of optimal policy). As discussed in Svensson and Woodford [26], the added term and the dependence on the Lagrange multiplier Ξ_{t-1} ensure that the minimization of (4.3), under either discretion or commitment, results in the optimal policy under commitment in a timeless perspective.[7]

The optimal policy choice, which results in the optimal policy projection, can now be formulized as choosing Y^t in the set of feasible projections in period t so as to minimize the modified intertemporal loss function, that is, to solve the problem

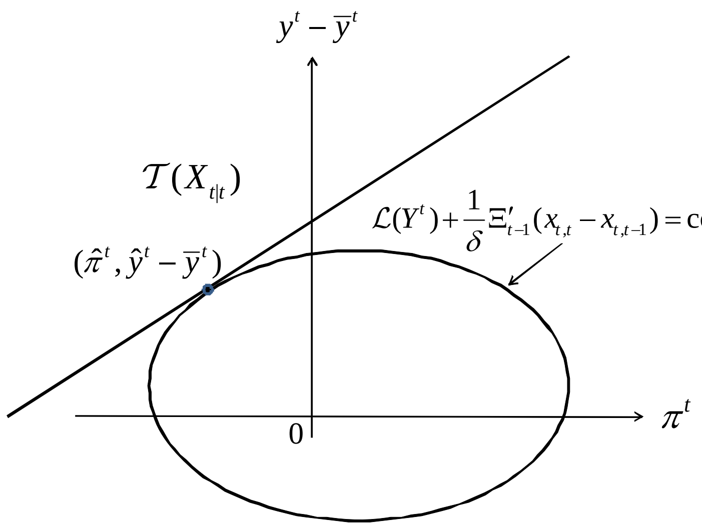
$$\min \mathcal{L}_t(Y^t) + \frac{1}{\delta}\Xi'_{t-1}(x_{t,t} - x_{t,t-1}) \text{ subject to } (X^t, x^t, i^t, Y^t) \in \mathcal{T}_t(X_{t|t}).$$

The set of feasible projections $\mathcal{T}_t(X_{t|t})$ is obviously very large and contains infinitely many different policy projections. The construction of alternative policy projections with alternative anticipated instrument-rate paths described in this paper can be seen as an attempt to narrow down the set of infinite alternative feasible policy projections to a finite number of alternatives for the policymaker to choose between.

For a given linear projection model and a given modified quadratic intertemporal loss function, it is possible to compute the optimal policy projection exactly. By varying the parameters of the

[7] This added term is closely related to the recursive saddlepoint method of Marcet and Marimon [17], see Svensson [24] for more discussion.

Figure 4.1: The set of feasible projections, an iso-loss curve, and the optimal policy choice



modified intertemporal loss function it is possible to generate alternative policy projections. Generating alternative policy projections in that way has the advantage that the policy projections are efficient in a specific sense. However, the policymaker may still prefer to see a few representative alternative policy projections constructed with alternative instrument-rate paths that are not constructed as optimal policy projections. The methods to construct policy projections for alternative anticipated instrument-rate paths presented in this paper makes that possible.

Figure 4.1 gives a simplified illustration of the optimal policy choice. The target variables are here inflation, π_t, and the output gap, $y_t - \bar{y}_t$, the gap between output, y_t, and potential output, $\bar{y}_t$, $Y_t \equiv (\pi_t,\, y_t - \bar{y}_t)'$. The inflation projection, π^t, is plotted along the horizontal axis, and the output-gap projection, $y^t - \bar{y}^t$, is plotted along the vertical axis. In the figure the inflation and output-gap projection are shown as one-dimensional, but we should think of them as multi-dimensional. Hence, a point in the figure corresponds to a point $(\pi^t, y^t - \bar{y}^t)$ in the multidimensional space of inflation and output-gap projections. The set of feasible projections, $\mathcal{T}(X_{t|t})$, is illustrated as the set of inflation and output-gap projections on and north-west of the positively sloped line. The location of the line depends on $X_{t|t}$, the estimated state of the economy in period t. In a special case, the quadratic intertemporal loss function can be illustrated by concentric ellipses, iso-loss curves, in the figure. The period loss function is

$$L_{t+\tau,t} = \begin{bmatrix} \pi_{t+\tau,t} \\ y_{t+\tau,t} - \bar{y}_{t+\tau,t} \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & \lambda_y \end{bmatrix} \begin{bmatrix} \pi_{t+\tau,t} \\ y_{t+\tau,t} - \bar{y}_{t+\tau,t} \end{bmatrix} = \pi_{t+\tau,t}^2 + \lambda(y_{t+\tau,t} - \bar{y}_{t+\tau,t})^2$$

for $\tau \geq 0$. The special case is when the forward-looking variables in the vector x_t only include current or expected future inflation rates and/or current or expected future output gaps. Then the added term, $\frac{1}{\delta}\Xi'_{t-1}(x_{t,t} - x_{t,t-1})$, in (4.3), only implies that the concentric ellipses need not necessarily be centered at the origin, which corresponds to the inflation projection and the output-gap projection both equal to zero. Iso-loss curves further out correspond to higher losses. The optimal policy projection, $(\hat{\pi}^t, \hat{y}^t - \bar{y}^t)$, is given by the tangency point between the set of feasible projection and an iso-loss curve. This is where the intertemporal loss (4.3) is minimized on the set of feasible projections.[8]

5. Optimal policy projections

The optimization results in a set of first-order conditions, which combined with the model equations (2.1) results in a system of difference equations (see Söderlind [21] and Svensson [24]). The system of difference equations can be solved with several alternative algorithms, for instance, those developed by Klein [13] and Sims [20] (see Svensson [23] and [24] for details of the derivation and the application of the Klein algorithm).[9]

Under the assumption of optimization under commitment in a timeless perspective, the optimal projection can be described by the following difference equation,

$$\begin{bmatrix} \hat{x}_{t+\tau,t} \\ \hat{\imath}_{t+\tau,t} \end{bmatrix} = \begin{bmatrix} F_x \\ F_i \end{bmatrix} \begin{bmatrix} \hat{X}_{t+\tau,t} \\ \Xi_{t+\tau-1,t} \end{bmatrix}, \quad \begin{bmatrix} \hat{X}_{t+\tau+1,t} \\ \Xi_{t+\tau,t} \end{bmatrix} = M \begin{bmatrix} \hat{X}_{t+\tau,t} \\ \Xi_{t+\tau-1,t} \end{bmatrix}, \tag{5.1}$$

$$\hat{Y}_{t+\tau,t} = D \begin{bmatrix} \hat{X}_{t+\tau,t} \\ \hat{x}_{t+\tau,t} \\ \hat{\imath}_{t+\tau,t} \end{bmatrix}, \tag{5.2}$$

for $\tau \geq 0$, where $\hat{X}_{t,t} = X_{t|t}$. The Klein algorithm returns the matrices F_x, F_i, and M. The submatrix F_i in (5.1) represents the optimal instrument rule,

$$i_{t,t+\tau} = F_i \begin{bmatrix} X_{t+\tau,t} \\ \Xi_{t+\tau-1,t} \end{bmatrix}. \tag{5.3}$$

[8] If there are other forward-looking variables than inflation and the output gap and these forward-looking variables have non-zero elements in the vector of Lagrange multipliers Ξ_{t-1}, the optimal policy choice cannot be illustrated in the space of only inflation and output-gap projections. Then these forward-looking variables require additional dimensions/axes in the illustration.

[9] The system of difference equations can also be solved with the so-called AIM algorithm of Anderson and Moore [9] and [10] (see Anderson [7] for a recent formulation). Whereas the Klein algorithm is easy to apply directly to the system of difference equations and the AIM algorithm requires some rewriting of the difference equations, the AIM algorithm appears to be significantly faster for large systems (see Anderson [8] for a comparison between AIM and other algorithms). The appendix of ALLS1 discusses the relation between the Klein and AIM algorithms and shows how the system of difference equations can be rewritten to fit the AIM algorithm.

These matrices depend on A, B, H, D, W, and δ, but they are independent of C. That they are independent of C demonstrates the certainty equivalence of optimal projections (the certainty equivalence that holds when the model is linear, the loss function is quadratic, and the shocks and the uncertainty are additive); only probability means of current and future variables are needed to determine optimal policy and the optimal projection. The n_X-vector $\Xi_{t+\tau,t}$ consists of the Lagrange multipliers of the lower block of (3.1), the block determining the projection of the forward-looking variables. The initial value for $\Xi_{t-1,t}$ is discussed in ALLS1.[10]

Alternative optimal projections can be constructed by varying the weights in the matrix W and the discount factor δ. Using alternative optimal projections has the advantage that the projections considered are efficient in the sense of minimizing an intertemporal loss function. That is, each projection is such that it is not possible to reduce the discounted sum of squared future projected deviations of a target variable from its target level without increasing the discounted sum of squared such future projected deviations of another target variable. In figure 4.1, the efficient subset of the set of feasible projections is given by the positively sloped line that is the boundary of the set of feasible projections. There are obvious advantages to restricting policy choices to be among efficient alternatives. Projections constructed with an arbitrary instrument rule (or with arbitrary deviations from an optimal instrument rule) are generally not efficient in this sense. That is, they correspond to points in the interior of the feasible set of projections, points north-west of the positively sloped line in figure 4.1.

In a backward-looking model, there are no forward-looking variables x_t, and in an optimal projection of a backward-looking variable, there are no Lagrange multipliers Ξ_t.

[10] The optimal projection can also be described as

$$\begin{bmatrix} \hat{X}_{t+\tau+1,t} \\ \Xi_{t+\tau,t} \\ \hat{x}_{t+\tau+1,t} \\ \hat{\imath}_{t+\tau+1,t} \end{bmatrix} = \bar{B} \begin{bmatrix} \hat{X}_{t+\tau,t} \\ \Xi_{t+\tau-1,t} \\ \hat{x}_{t+\tau,t} \\ \hat{\imath}_{t+\tau,t} \end{bmatrix}, \tag{5.4}$$

for $\tau \geq 0$, where

$$\begin{aligned} \hat{X}_{t,t} &= X_{t|t}, \\ \Xi_{t-1,t} &= \Xi_{t-1,t-1}, \\ \begin{bmatrix} \hat{x}_{t,t} \\ \hat{\imath}_{t,t} \end{bmatrix} &= \begin{bmatrix} \bar{B}_{x\cdot} \\ \bar{B}_{i\cdot} \end{bmatrix} \begin{bmatrix} \hat{X}_{t-1,t-1} \\ \Xi_{t-2,t-1} \\ \hat{x}_{t-1,t-1} \\ \hat{\imath}_{t-1,t-1} \end{bmatrix} + \begin{bmatrix} \Phi_{x\cdot} \\ \Phi_{i\cdot} \end{bmatrix} \begin{bmatrix} X_{t|t} - \hat{X}_{t,t-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

where the $\bar{B}$ and Φ are matrices returned by the AIM algorithm and partitioned conformably with X_t, Ξ_{t-1}, x_t, and i_t.(The notation $\Phi_{x\cdot}$ refers to the larger sub matrix $[\Phi_{xX}\ \Phi_{x\Xi}\ \Phi_{xx}\ \Phi_{xi}]$; Φ_{xX} is the leftmost submatrix of this submatrix.) Here, $X_{t|t} - \hat{X}_{t,t-1}$ denotes the difference between the estimate $X_{t|t}$, which is based on information available in the beginning of period t, and the projection $\hat{X}_{t,t-1}$, which is based on the information available in the beginning of period $t-1$.

6. Projections with a constant arbitrary instrument rule

With a constant (that is, time-invariant) arbitrary instrument rule, the instrument rate satisfies

$$i_t = [f_X \quad f_x] \begin{bmatrix} X_t \\ x_t \end{bmatrix} \tag{6.1}$$

for $\tau \geq 0$, where the $n_i \times (n_X + n_x)$ matrix $[f_X \quad f_x]$ is a given (linear) instrument rule and partitioned conformably with X_t and x_t. If $f_x \equiv 0$, the instrument rule is an *explicit* instrument rule; if $f_x \neq 0$, the instrument rule is an *implicit* instrument rule. In the latter case, the instrument rule is actually an equilibrium condition, in the sense that in a real-time analogue the instrument rate in period t and the forward-looking variables in period t would be simultaneously determined.

The instrument rule that is estimated for Ramses is of the form (see the appendix of ALLS1 for the notation)

$$\begin{aligned} i_t \;=\; & \rho_R i_{t-1} + (1-\rho_R)\left[\hat{\bar{\pi}}_t^c + r_\pi\left(\hat{\pi}_{t-1}^c - \hat{\bar{\pi}}_t^c\right) + r_y \hat{y}_{t-1} + r_x \hat{\bar{x}}_{t-1}\right] \\ & + r_{\Delta\pi}\left(\hat{\pi}_t^c - \hat{\pi}_{t-1}^c\right) + r_{\Delta y}\left(\hat{y}_t - \hat{y}_{t-1}\right) + \varepsilon_{Rt}, \end{aligned} \tag{6.2}$$

where $i_t \equiv \hat{R}_t$ (the notation for the short nominal interest rate in Ramses). Since $\hat{\pi}_t^c$ and $\hat{y}_t$, the deviation of CPI inflation and output from trend, are forward-looking variables in Ramses, this is an implicit instrument rule.

An arbitrary more general (linear) policy rule (G, f) can be written as

$$G_x x_{t+1|t} + G_i i_{t+1|t} = f_X X_t + f_x x_t + f_i i_t, \tag{6.3}$$

where the $n_i \times (n_x + n_i)$ matrix $G \equiv [G_x \quad G_i]$ is partitioned conformably with x_t and i_t and the $n_i \times (n_X + n_x + n_i)$ matrix $f \equiv [f_X \quad f_x \quad f_i]$ is partitioned conformably with X_t, x_t, and i_t. This general policy rules includes explicit, implicit, and forecast-based instrument rules (in the latter the instrument rate depends on expectations of future forward-looking variables, $x_{t+1|t}$) as well as targeting rules (conditions on current or expected future target variables).[11] When this general policy rule is an instrument rule, we require the $n_x \times n_i$ matrix f_i to be nonsingular, so (6.3) determines i_t for given X_t, x_t, $x_{t+1|t}$, and $i_{t+1|t}$.

The optimal instrument rule under commitment can be written on the form

$$0 = F_{iX} X_t + F_{i\Xi} \Xi_{t-1} - i_t, \tag{6.4}$$

[11] A targeting rule can be expressed in terms of expected leads, current values, and lags of the target variables; see Svensson [22] and Svensson and Woodford [26].

where the matrix F_i in (5.3) is partitioned conformably with X_t and Ξ_{t-1}. Here the n_x-vector of Lagrange multipliers Ξ_t in equilibrium follows

$$\Xi_t = M_{\Xi X} X_t + M_{\Xi\Xi} \Xi_{t-1}, \tag{6.5}$$

where the matrices M in (5.2) has been portioned conformably with X_t, and Ξ_{t-1}. Thus, in order to include this optimal instrument rule in the set of policy rules (6.3) considered, the predetermined variables need to be augmented with Ξ_{t-1} and the equations for the predetermined variables with (6.5). For simplicity, the treatment below does not include this augmentation. Alternatively, below the vector of predetermined variables could consistently be augmented with the vector of Lagrange multipliers, so everywhere we would have $(X_t', \Xi_{t-1}')'$ instead of X_t, with corresponding augmentation of the relevant matrices.

The general policy rule can be added to the model equations (2.1) to form the new system to be solved. With the notation $\tilde{x}_t \equiv (x_t', i_t')'$, the new system can be written

$$\begin{bmatrix} X_{t+1} \\ \tilde{H}\tilde{x}_{t+1|t} \end{bmatrix} = \tilde{A} \begin{bmatrix} X_t \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} C \\ 0_{(n_x+n_i)\times n_\varepsilon} \end{bmatrix} \varepsilon_{t+1}, \tag{6.6}$$

$$Y_t = D \begin{bmatrix} X_t \\ \tilde{x}_t \end{bmatrix},$$

where

$$\tilde{H} \equiv \begin{bmatrix} H & 0 \\ G_x & G_i \end{bmatrix}, \quad \tilde{A} \equiv \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ f_X & f_x & f_i \end{bmatrix},$$

where $\tilde{H}$ is partitioned conformably with x_t and i_t and $\tilde{A}$ is partitioned conformably with X_t, x_t, and i_t. The corresponding projection model can then be written

$$\begin{bmatrix} X_{t+\tau+1,t} \\ \tilde{H}\tilde{x}_{t+\tau+1,t} \end{bmatrix} = \tilde{A} \begin{bmatrix} X_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix},$$

$$Y_{t+\tau,t} = D \begin{bmatrix} X_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix},$$

for $\tau \geq 0$, where $X_{t,t} = X_{t|t}$.

Then, under the assumption that the policy rule gives rise to the saddlepoint property (that the number eigenvalues with modulus greater than unity is equal to the number of non-predetermined variables) there exist matrices M and F such that the resulting *equilibrium projection* satisfies

$$X_{t+\tau+1,t} = M X_{t+\tau,t},$$

$$\tilde{x}_{t+\tau,t} \equiv \begin{bmatrix} x_{t+\tau,t} \\ i_{t+\tau,t} \end{bmatrix} = F X_{t+\tau,t} \equiv \begin{bmatrix} F_x \\ F_i \end{bmatrix} X_{t+\tau,t}$$

for $\tau \geq 0$, where the matrices M and F depend on $\tilde{A}$ and $\tilde{H}$, and thereby on A, B, H, G, and f.[12]

In a backward-looking model, the time-invariant instrument rule depends on the vector of predetermined variables only, since there are no forward-looking variables, and the vector $\tilde{x}_t$ is identical to i_t.

7. Projections with time-varying restrictions on the instrument rate

Consider a restriction on the instrument-rate projection of the form

$$i_{t+\tau,t} = \bar{\imath}_{t+\tau,t}, \quad \tau = 0, ..., T, \tag{7.1}$$

where $\{\bar{\imath}_{t+\tau,t}\}_{\tau=0}^{T}$ is a sequence of $T+1$ given instrument-rate levels. Alternatively, we can have restriction on the real instrument-rate projection of the form

$$r_{t+\tau,t} = \bar{r}_{t+\tau,t}, \quad \tau = 0, ..., T, \tag{7.2}$$

where

$$r_t \equiv i_t - \pi_{t+1|t} \tag{7.3}$$

is the real instrument rate and $\pi_{t+1|t}$ is expected inflation. With restrictions of this kind, the nominal or real instrument rate is exogenous for period $0, 1, ..., T$.

These restrictions are anticipated by both the central bank and the private sector, in contrast to Leeper and Zha [14] where they are anticipated and planned by the central bank but not anticipated by the private sector. Thus, the current case corresponds to a situation where the restriction is announced to the private sector by the central bank and believed by the private sector, whereas the Leeper and Zha case corresponds to a situation where the central bank either makes secret plans to implement the restriction or the restriction is announced but not believed by the private sector.

The restrictions are followed by an anticipated switch in period $T+1$ to the policy rule (G, f) in order to guarantee determinacy. The restrictions can be implemented by adding time-varying

[12] Equivalently, the resulting equilibrium projection satisfies

$$\begin{bmatrix} \hat{X}_{t+\tau+1,t} \\ \hat{x}_{t+\tau+1,t} \\ \hat{\imath}_{t+\tau+1,t} \end{bmatrix} = \bar{B} \begin{bmatrix} \hat{X}_{t+\tau,t} \\ \hat{x}_{t+\tau,t} \\ \hat{\imath}_{t+\tau,t} \end{bmatrix}$$

for $\tau \geq 0$, where

$$\begin{aligned} \hat{X}_{t,t} &= X_{t|t}, \\ \begin{bmatrix} \hat{x}_{t,t} \\ \hat{\imath}_{t,t} \end{bmatrix} &= \begin{bmatrix} \bar{B}_x \\ \bar{B}_i \end{bmatrix} \begin{bmatrix} \hat{X}_{t-1,t-1} \\ \hat{x}_{t-1,t-1} \\ \hat{\imath}_{t-1,,t-1} \end{bmatrix} + \begin{bmatrix} \Phi_{xX} \\ \Phi_{iX} \end{bmatrix} (X_{t|t} - \hat{X}_{t,t-1}). \end{aligned}$$

constants to the policy rule (G, f),

$$G_x x_{t+\tau+1,t} + G_i i_{t+\tau+1,t} = f_X X_{t+\tau,t} + f_x x_{t+\tau,t} + f_i i_{t+\tau,t} + z_{t+\tau,t}, \quad \tau = 0, ..., T, \tag{7.4}$$

where the sequence of scalars $\{z_{t+\tau,t}\}_{\tau=0}^{T}$ is chosen such that (7.1) or (7.2) is satisfied.

More precisely, we let the $(T+1)$-vector $z^t \equiv (z_{t,t}, z_{t+1,t}, ..., z_{t+T,t})'$ denote a projection of the stochastic process $\{z_{t+\tau}\}_{\tau=0}^{T}$ of the stochastic variable $z_{t+\tau}$. The stochastic variable z_t is called the deviation, as in the treatment of central-bank judgment in Svensson [23]. In particular, we assume that the deviation satisfies

$$z_t = \eta_{t,t} + \sum_{s=1}^{T} \eta_{t,t-s}$$

for a given $T \geq 0$, where $\eta^t \equiv (\eta'_{t,t}, \eta'_{t+1,t}, ..., \eta'_{t+T,t})'$ is a zero-mean i.i.d. random $(T+1)$-vector realized in the beginning of period t and called the innovation in period t. For $T = 0$, $z_t = \eta_t$ is then a simple i.i.d. disturbance. For $T > 0$, the deviation instead follows a moving-average process. Then we have

$$\begin{aligned} z_{t+\tau,t+1} &= z_{t+\tau,t} + \eta_{t+\tau,t+1}, \quad \tau = 1, ..., T, \\ z_{t+\tau+T+1,t+1} &= \eta_{t+T+1,t+1}. \end{aligned}$$

It follows that the dynamics of the deviation can be written

$$z^{t+1} = A_z z^t + \eta^{t+1}, \tag{7.5}$$

where the $(T+1) \times (T+1)$ matrix A_z is defined as

$$A_z \equiv \begin{bmatrix} 0_{T\times1} & I_T \\ 0 & 0_{1\times T} \end{bmatrix}.$$

Hence, z^t is the central bank's mean projection of current and future deviations, and η^t can be interpreted as the new information the central bank receives in the beginning of period t about those deviations.[13]

Let us now combine the projection model, (3.1), with the restriction (7.4) and the dynamics of the deviation, (7.5). We can then write the combined projection model as

$$\begin{bmatrix} \tilde{X}_{t+\tau+1,t} \\ \tilde{H}\tilde{x}_{t+\tau+1,t} \end{bmatrix} = \tilde{A} \begin{bmatrix} \tilde{X}_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix} \tag{7.6}$$

[13] In Svensson [23] the deviation z_t is an n_z-vector of terms entering the different equations in the model, and the projection z^t of future z_t deviation is identified with central-bank judgment. The graphs in Svensson [23] can be seen as impulse responses to η^t, the new information about future deviations. (The notation here is slightly different from Svensson [23] in that there the projection z^t does not include the current deviation.)

for $\tau \geq 0$, where

$$\tilde{X}_t \equiv \begin{bmatrix} X_t \\ z^t \end{bmatrix}, \quad \tilde{x}_t \equiv \begin{bmatrix} x_t \\ i_t \end{bmatrix}, \quad \tilde{H} \equiv \begin{bmatrix} H & 0 \\ G_x & G_i \end{bmatrix},$$

$$\tilde{A} \equiv \begin{bmatrix} A_{11} & 0_{n_X \times 1} & 0_{n_X \times T} & A_{12} & B_1 \\ 0_{T \times n_X} & 0_{T \times 1} & I_T & 0_{T \times n_x} & 0_{T \times 1} \\ 0_{1 \times n_X} & 0 & 0_{1 \times T} & 0_{1 \times n_x} & 0 \\ A_{21} & 0_{n_x \times 1} & 0_{n_x \times T} & A_{22} & B_2 \\ f_X & 1 & 0_{1 \times T} & f_x & f_i \end{bmatrix}.$$

Under the assumption of the saddlepoint property, the system of difference equations (7.6) has a unique solution and there exist unique matrices M and F such that the solution can be written

$$\begin{aligned} \tilde{X}_{t+\tau,t} &= M^\tau \tilde{X}_{t,t}, \\ \tilde{x}_{t+\tau,t} &= F\tilde{X}_{t+\tau,t} = FM^\tau \tilde{X}_{t,t} \end{aligned}$$

for $\tau \geq 0$, where $X_{t,t}$ in $\tilde{X}_{t,t} \equiv (X'_{t,t}, z^{t\prime})'$ is given but the $(T+1)$-vector z^t remains to be determined. Its elements are then determined by the restrictions (7.1) or (7.2).

In order to satisfy the restriction (7.1) on the nominal instrument rate, we note that it can now be written

$$i_{t+\tau,t} = F_i M^\tau \begin{bmatrix} X_{t,t} \\ z^t \end{bmatrix} = \bar{\imath}_{t+\tau,t}, \quad \tau = 0, 1, ..., T.$$

This provides $T+1$ linear equations for the $T+1$ elements of z^t.

In order to instead satisfy the restriction (7.2) on the real instrument rate, we note that inflation expectations in a DSGE model similar to Ramses generally satisfy

$$\pi_{t+1|t} \equiv \varphi \tilde{x}_{t+1|t} + \Phi \begin{bmatrix} \tilde{X}_t \\ \tilde{x}_t \end{bmatrix}. \tag{7.7}$$

for some vectors φ and Φ. For instance, if π_t is one of the elements of x_t, the corresponding element of φ is unity, all other elements of φ are zero, and $\Phi \equiv 0$. If $\pi_{t+1|t}$ is one of the elements of $\tilde{x}_t$, the corresponding element of Φ is unity, all other elements of Φ are zero, and $\varphi \equiv 0$.[14] Then the restriction (7.2) can be written

$$r_{t+\tau,t} \equiv i_{t+\tau,t} - \pi_{t+\tau+1,t} = (F_i - \varphi FM - \Phi) M^\tau \begin{bmatrix} X_{t,t} \\ z^t \end{bmatrix} = \bar{r}_{t+\tau,t}, \quad \tau = 0, 1, ..., T.$$

This again provides $T+1$ linear equations for the $T+1$ elements of z^t.

[14] Note that φ and Φ here are structural, not reduced-form expressions.

7.1. Backward-looking model

In a backward-looking model, the projection of the instrument rule with the time-varying constraints can be written

$$i_{t+\tau,t} = f_X X_{t+\tau,t} + z_{t+\tau,t}, \tag{7.8}$$

so it is trivial to determine the projection z^t recursively so as to satisfy the restriction (7.1) on the nominal instrument-rate projection. Inflation can be written

$$\pi_t = \Phi X_t$$

for some vector Φ, so expected inflation can be written

$$\pi_{t+1|t} = \Phi X_{t+1|t} = \Phi(AX_t + Bi_t). \tag{7.9}$$

By combining (7.8), (7.9) and (7.3), it is trivial to determine the projection z^t so as to satisfy the restriction (7.2) on the real instrument-rate projection.

8. Examples

In this section we examine restrictions on the nominal and real instrument-rate path for the backward-looking Rudebusch-Svensson model and the two forward-looking models, the Lindé model and Ramses. Appendices C provide some details on the Rudebusch-Svensson and Lindé models. We also show a simulation with Ramses with the method of modest interventions by Leeper and Zha. Appendix A provides some details on the Leeper-Zha method.

8.1. The Rudebusch-Svensson model

The backward-looking empirical Rudebusch-Svensson model [18] has two equations (with estimates rounded to two decimal points)

$$\pi_{t+1} = 0.70\,\pi_t - 0.10\,\pi_{t-1} + 0.28\,\pi_{t-2} + 0.12\,\pi_{t-3} + 0.14\,y_t + \varepsilon_{\pi,t+1}, \tag{8.1}$$

$$y_{t+1} = 1.16\,y_t - 0.25\,y_{t-1} - 0.10\left(\frac{1}{4}\Sigma_{j=0}^{3} i_{t-j} - \frac{1}{4}\Sigma_{j=0}^{3}\pi_{t-j}\right) + \varepsilon_{y,t+1}. \tag{8.2}$$

The period is a quarter, and π_t is quarterly GDP inflation measured in percentage points at an annual rate, y_t is the output gap measured in percentage points, and i_t is the quarterly average of the federal-funds rate, measured in percentage points at an annual rate. All variables are measured as differences from their means, their steady-state levels. The predetermined variables are $X_t \equiv (\pi_t, \pi_{t-1}, \pi_{t-2,}\pi_{t-2}, y_t, y_{t-1}, i_{t-1}, i_{t-2,}i_{t-3})'$. See appendix C for details.

The target variables are inflation, the output gap, and the first-difference of the federal funds rate. The period loss function is

$$L_t = \frac{1}{2}[\pi_t^2 + \lambda_y y_t^2 + \lambda_{\Delta i}(i_t - i_{t-1})^2], \tag{8.3}$$

where π_t is measured as the difference from the inflation target, which is equal to the steady-state level. The discount factor, δ, and the relative weights on the output-gap stabilization, λ_y, and interest-rate smoothing, $\lambda_{\Delta i}$, are set to satisfy $\delta = 1$, $\lambda_y = 1$, and $\lambda_{\Delta i} = 0.2$.

For the loss function (8.3) with the parameters $\delta = 1$, $\lambda_y = 1$, and $\lambda_{\Delta i} = 0.2$, and the case where ε_t is an i.i.d. shock with zero mean, the optimal instrument rule is (the coefficients are rounded to two decimal points)

$$i_t = 1.22\,\pi_t + 0.43\,\pi_{t-1} + 0.53\,\pi_{t-2} + 0.18\,\pi_{t-3} + 1.93\,y_t - 0.49\,y_{t-1} + 0.36\,i_{t-1} - 0.09\,i_{t-2} - 0.05\,i_{t-3}.$$

Figure 8.1 shows projections for the Rudebusch-Svensson model. The top row of panels show projections under the optimal policy, whereas the bottom row of panels show projections under a Taylor rule,

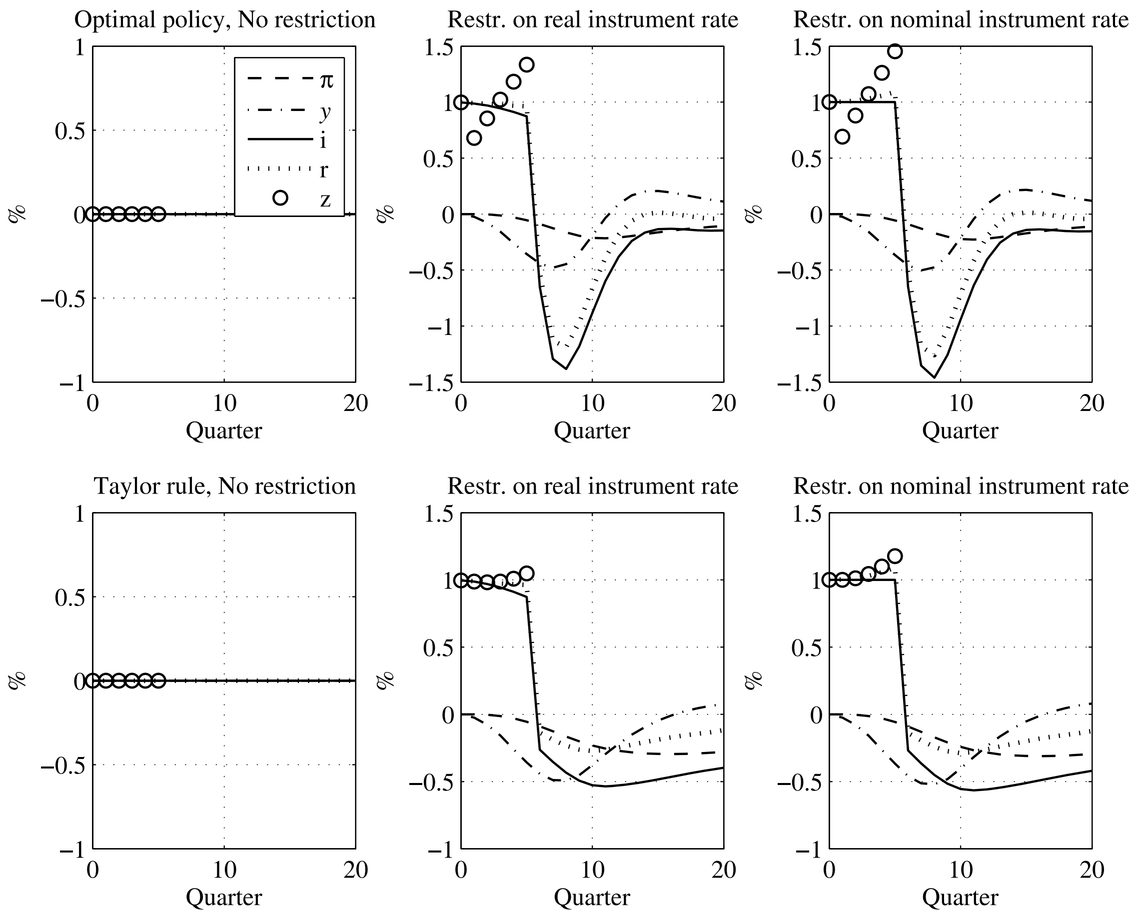
$$i_t = 1.5\,\pi_t + 0.5\,y_t,$$

where the instrument responds to the predetermined inflation and output gap with the standard coefficients 1.5 and 0.5, respectively.

The projections start in quarter 0 from the steady state, when all the predetermined variables are zero. The left column of panels show the projections when there is no restriction imposed on the nominal or real instrument-rate path. This correspond to zero constants $z_{t+\tau,t}$ in the optimal instrument rule and the Taylor rule. These are denoted by circles for the first six quarters, quarters 0–5. The economy remains in the steady state, and inflation (denoted by a dashed curve), the output gap (denoted by a dashed-dotted curve), the nominal instrument rate (denoted by a solid curve), and the real instrument rate (denoted by a dotted curve) all remain at zero.

The right column shows projections when the nominal instrument-rate is restricted to equal unity for the first six quarters. For both optimal policy and the Taylor rule, this requires positive and increasing time-varying constants in the instrument rule. The upward shift in quarters 0–5 in the nominal instrument-rate path reduces inflation and expected inflation somewhat, and the real instrument rate path shifts up a bit more than the nominal instrument-rate path. The increased

Figure 8.1: Projections for Rudebusch-Svensson model with unrestricted and restricted real and nominal interest rate for optimal policy (top row) and Taylor rule (bottom row): 6-quarter restriction



real instrument rate also reduces the output gap. In the Rudebusch-Svensson model, inflation is very sluggish and the output gap responds more to the nominal and real instrument rate than inflation. From quarter 6, there is no restriction on the instrument-rate path, and according to both the optimal policy and the Taylor rule, the nominal and real instrument rate are reduced substantially so as to bring the negative inflation and output gap eventually back to zero. The optimal policy is more effective in bringing back inflation and the output gap than the Taylor rule, which is natural since the Taylor rule is not optimal.

The middle column shows projections when the real instrument rate is restricted to equal unity during quarters 0–5. Since there is so little movement in inflation and expected inflation, the projections for these restrictions on the real and the nominal instrument rate are very similar.

Since there are no forward-looking variables in the Rudebusch-Svensson model, there would be no difference between these projections with anticipated restrictions on the instrument-rate path and simulations with unanticipated shocks as in Leeper and Zha [14].

8.2. The Lindé model

The empirical New Keynesian model of the US economy due to Lindé [15] also has two equations. We use the following parameter estimates,

$$\begin{aligned}\pi_t &= 0.457\,\pi_{t+1|t} + (1-0.457)\pi_{t-1} + 0.048 y_t + \varepsilon_{\pi t},\\ y_t &= 0.425\,y_{t+1|t} + (1-0.425)y_{t-1} - 0.156(i_t - \pi_{t+1|t}) + \varepsilon_{yt}.\end{aligned}$$

The period is a quarter, and π_t is quarterly GDP inflation measured in percentage points at an annual rate, y_t is the output gap measured in percentage points, and i_t is the quarterly average of the federal-funds rate, measured in percentage points at an annual rate. All variables are measured as differences from their means, their steady-state levels. The shock $\varepsilon_t \equiv (\varepsilon_{\pi t}, \varepsilon_{yt})'$ is i.i.d. with mean zero.

For the loss function (8.3), the predetermined variables are $(\varepsilon_{\pi t}, \varepsilon_{yt}, \pi_{t-1}, y_{t-1}, i_{t-1})$ (the lagged instrument rate enters because it enters into the loss function, and the two shocks are included among the predetermined variables in order to write the model on the form (2.1) with no shocks in the equations for the forward-looking variables). The forward-looking variables are (π_t, y_t). See appendix D for details.[15]

For the loss function (8.3) with the parameters $\delta = 1$, $\lambda_y = 1$, and $\lambda_{\Delta i} = 0.2$, the optimal policy function (6.4) is (the coefficients are rounded to two decimal points),

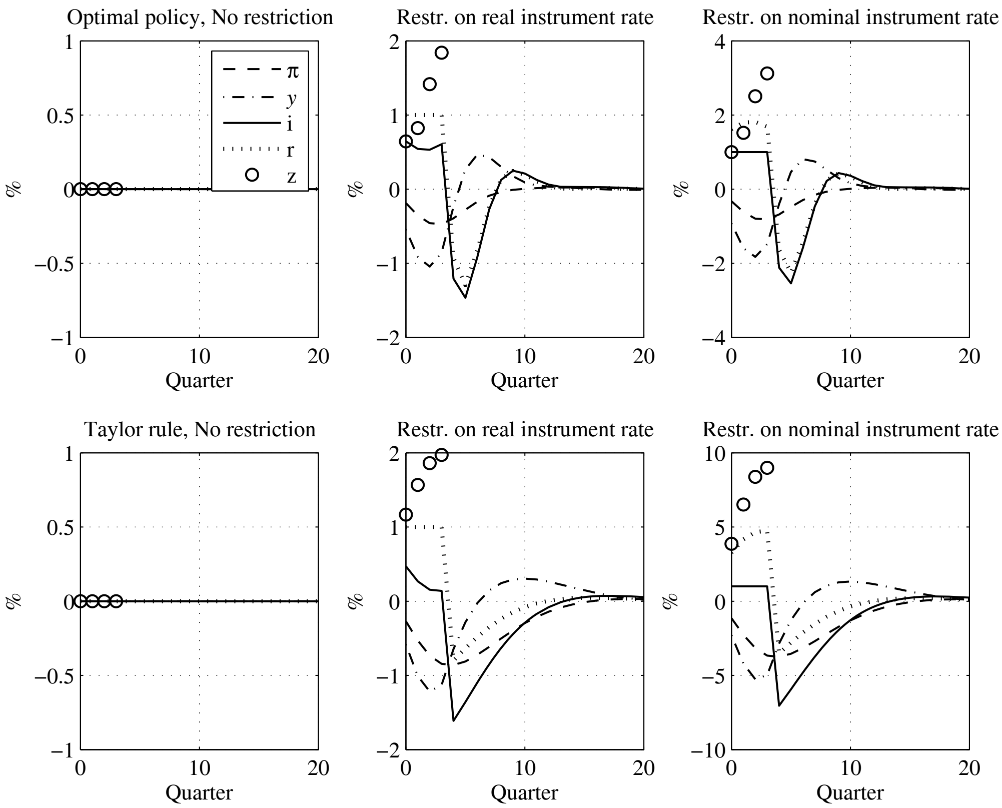
$$i_t = 1.06\,\varepsilon_{\pi t} + 1.38\,\varepsilon_{yt} + 0.58\,\pi_{t-1} + 0.78\,y_{t-1} + 0.40\ i_{t-1} + 0.02\,\Xi_{\pi,t-1,t-1} + 0.20\,\Xi_{y,t-1,t-1},$$

where $\Xi_{\pi,t-1,t-1}$ and $\Xi_{y,t-1,t-1}$ are the Lagrange multipliers for the two equations for the forward-looking variables in the decision problem in period $t-1$. The difference equation (6.5) for the Lagrange multipliers is

$$\begin{bmatrix} \Xi_{\pi t} \\ \Xi_{yt} \end{bmatrix} = \begin{bmatrix} 10.20 & 0.74 & 5.54 & 0.43 & -0.21 \\ 0.74 & 1.48 & 0.40 & 0.85 & -0.28 \end{bmatrix} \begin{bmatrix} \varepsilon_{\pi t} \\ \varepsilon_{yt} \\ \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} 0.72 & 0.16 \\ 0.03 & 0.38 \end{bmatrix} \begin{bmatrix} \Xi_{\pi,t-1} \\ \Xi_{y,t-1} \end{bmatrix}.$$

[15] It is arguably unrealistic to consider inflation and output in the current quarter as forward-looking variables. Alternataively, current inflation and the output gap could be treated as predetermined, and one-quarter-ahead inflation, output-gap, and instrument-rate plans could be determined by the model above. Such a variant of the newkeynesian model is used in Svensson and Woodford [26].

Figure 8.2: Projections for the Lindé model with unrestricted and restricted real and nominal interest rate for optimal policy (top row) and Taylor rule (bottom row): 4-quarter restriction

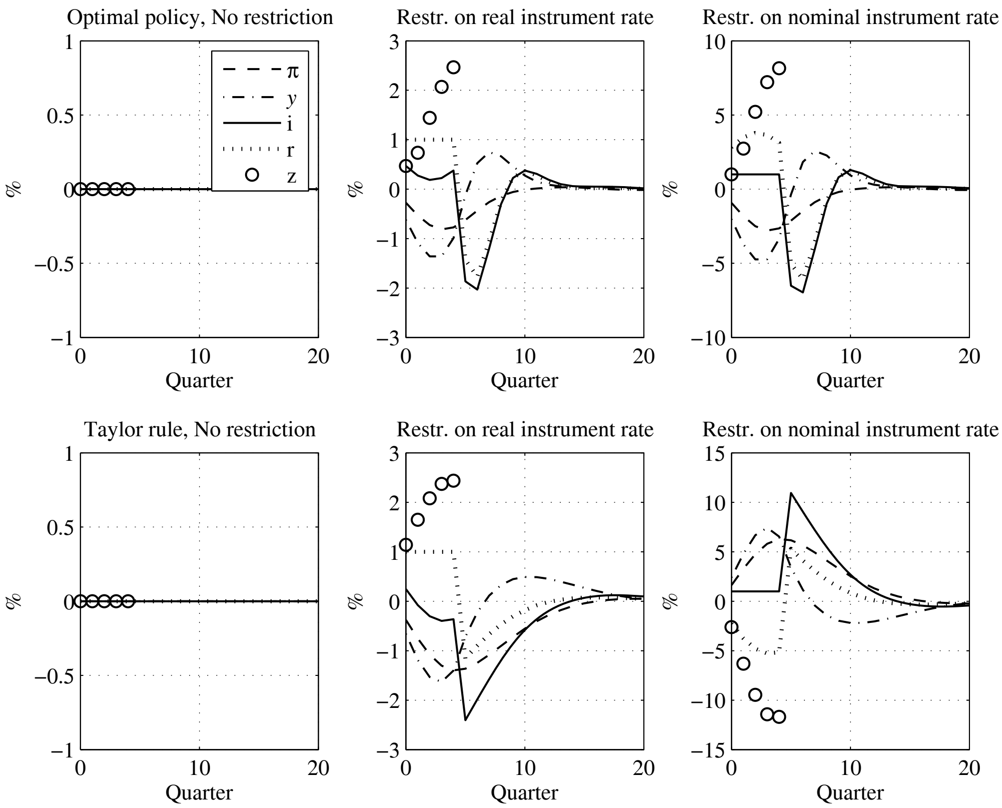


We also examine the projections for a Taylor rule for which the instrument rate responds to the forward-looking current inflation and output,

$$i_t = 1.5\,\pi_t + 0.5\,y_t.$$

Figure 8.2 shows projections for the optimal policy (top row) and Taylor rule (bottom row) when there is a restriction to equal unity for quarters 0–3 for the real instrument rate (middle column) and the nominal instrument rate (right column). In the middle column, the restriction on the real instrument rate reduces inflation quite a bit, and the corresponding nominal instrument-rate projection is quite a bit below unity and the real instrument rate for quarters 0–3, but still positive. In line with this, in the right column, the restriction on the nominal instrument rate to

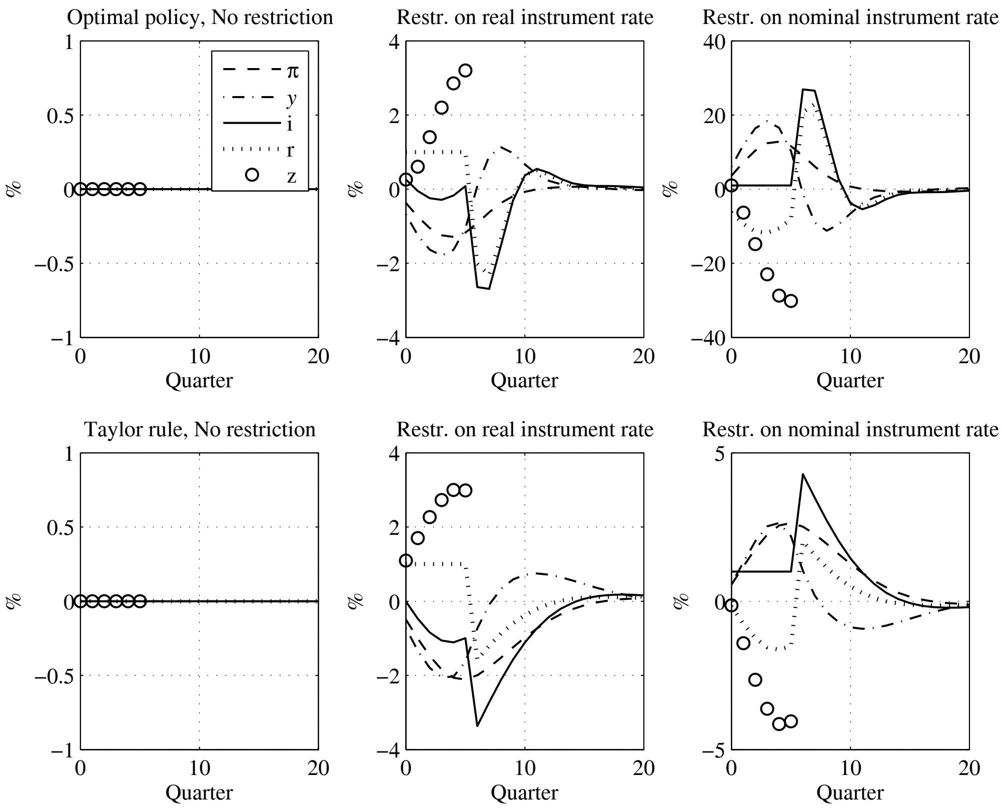
Figure 8.3: Projections for the Lindé model with unrestricted and restricted real and nominal interest rate for optimal policy (top row) and Taylor rule (bottom row): 5-quarter restriction



equal unity for quarters 0–3 implies that the real instrument-rate projection is higher than unity for those quarters. We note that these restrictions require positive and rapidly rising time-varying constants (denoted by the circles). The magnitude of the constants are larger than those in figure 8.1 for the Rudebusch-Svensson model. Using the magnitude of the constant as indicating the severity of the restriction, we conclude that the restriction to nominal or real instrument rates equal to unity is more sever in the Lindé model, also when we only impose those restrictions for three quarters in figure 8.2 instead of six quarters in figure 8.1.

Because inflation is more sensitive to movements in the real instrument rate in the Lindé model than in the Rudebusch-Svensson model, there is a greater difference between restrictions on the

Figure 8.4: Projections for the Lindé model with unrestricted and restricted real and nominal interest rate for optimal policy (top row) and Taylor rule (bottom row): 6-quarter restriction



nominal and the real instrument rate. Also, from quarter 4, when there is no restriction on the instrument rate, a fall in the real and nominal real instrument rate, according to both the optimal policy and the Taylor rule, more easily stabilizes inflation and the output gap back to the steady state than in the Rudebusch-Svensson model.

In figure 8.3, we extend the restrictions on the nominal and real instrument rate another quarter, to quarter 4. If we look at the panel in the bottom row, middle column, for the Taylor rule and the restriction on the real instrument rate, we see that the shift up of the real instrument rate during quarters 0–5 now shifts the inflation projection and inflation expectations down so much that the corresponding nominal instrument-rate projection becomes negative. If we nevertheless in the bottom right panel insists on shifting up the nominal instrument-rate path and restricting the nominal instrument rate to become positive and equal to unity for quarters 0–4, we see that there is then no equilibrium with a positive real instrument rate and lower inflation during quarters 0–4. Instead, there is a very different equilibrium with high inflation and a negative real interest-rate during quarters 0–4. Thus, imposing a restriction on the nominal instrument rate this way gives way to a very different equilibrium than we first might have expected.

As we can see in the top row in figure 8.3, this phenomenon does not happen under the optimal policy. There, the restriction on the nominal instrument-rate to equal unity for quarters 0–4 is consistent with a higher real instrument rate and a lower, but not too low, inflation rate. The difference between these projections in the right column of the figure is of course because expectations of optimal policy from quarter 5 rather than the Taylor rule implies more stability in inflation before quarter 5.

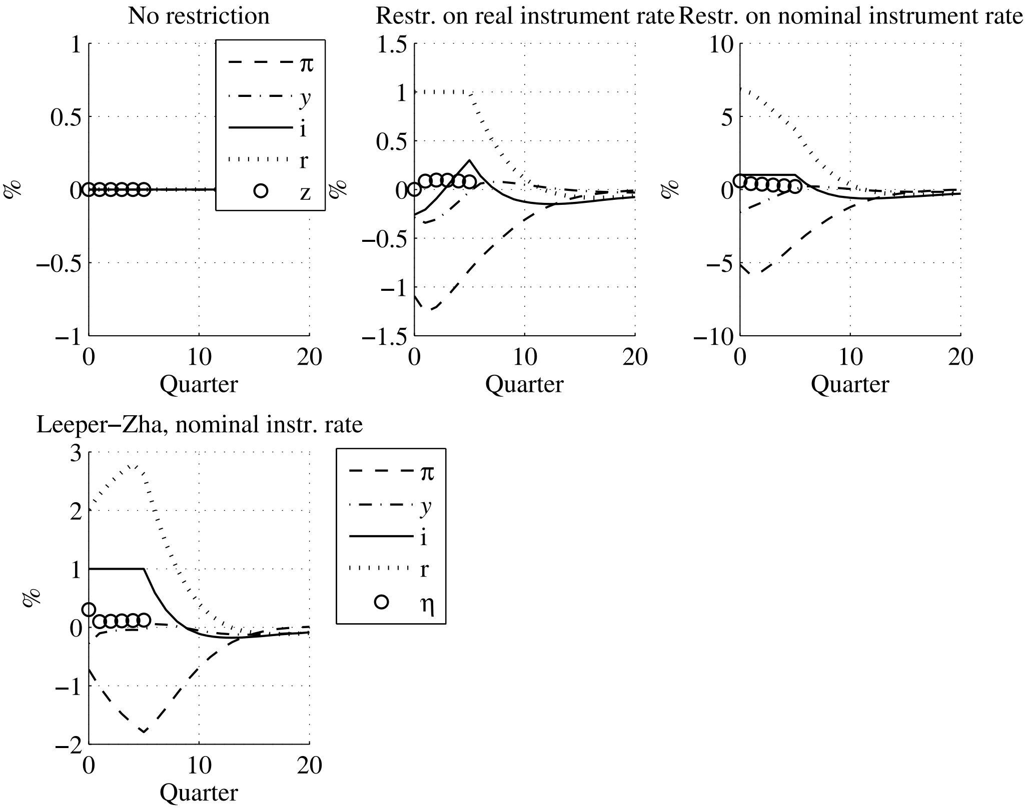
In figure 8.4, we extend the restrictions to six quarters, quarters 0–5. Then we see that the phenomenon of high inflation and low real instrument rate for the restrictions on the nominal instrument rate (right column) occurs for both the optimal policy and the Taylor rule.

We see from these examples that high sensitivity of the inflation projection and inflation expectations to changes in the real instrument rate may cause unusual equilibria when restrictions on the nominal instrument rate is imposed for sufficiently many quarters. Our conclusion is that it is more robust to impose restrictions on the real instrument rate. Also, this makes sense from the point of view that it is real interest rates that matter for output and inflation.

8.3. Ramses

The appendix of ALLS1 provides more details on Ramses, including the elements of the vectors X_t, x_t, i_t, and ε_t. Figure 8.5 shows projections in Ramses for the estimated instrument rule. The top row shows the result of restrictions on the nominal and real instrument rate for six quarters, quarters 0–5. We see that the phenomenon with a shift to a high inflation projection and low real instrument-rate projection that occurred for restrictions on a positive nominal instrument rate for the Lindé model does not occur for this length of restrictions in Ramses. However, we can see that there is a substantial difference between restrictions on the nominal and the real instrument rate. In the top middle panel we see that the restriction on the real instrument rate to equal unity for quarters 0–5 corresponds to a nominal instrument-projection that starts out negative and then become positive. In the top right panel, we see that a restriction on the nominal instrument-rate

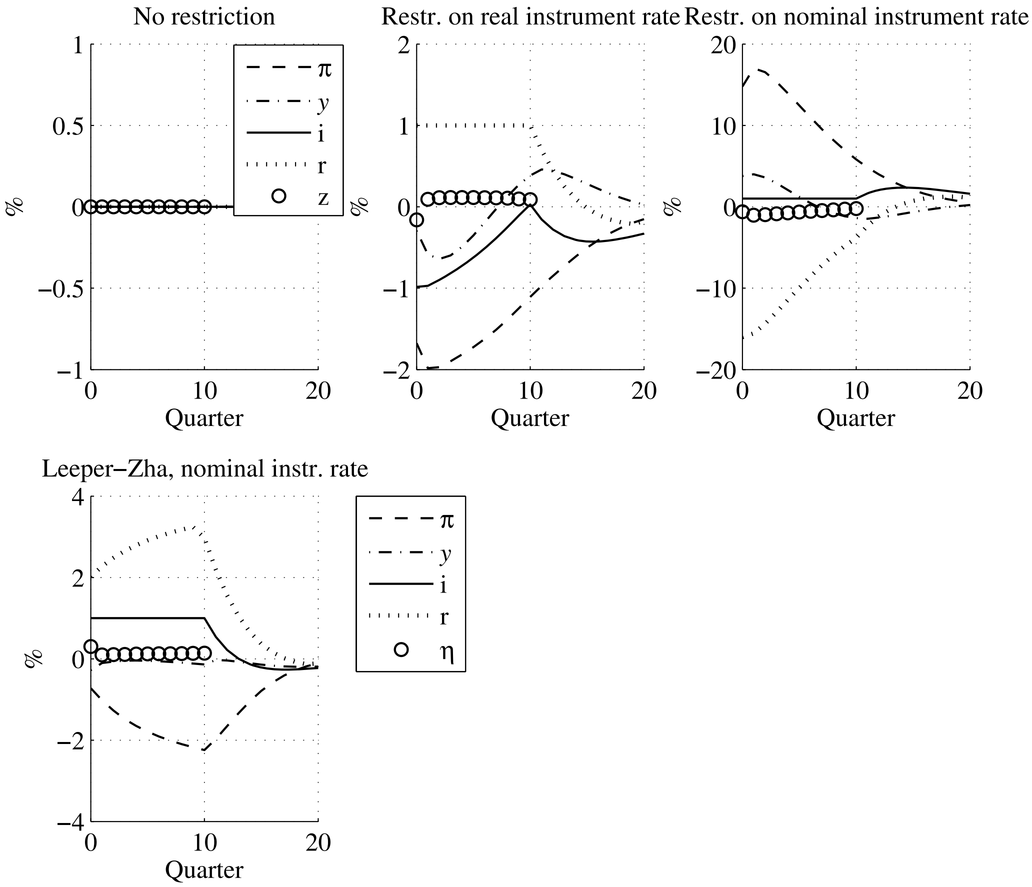
Figure 8.5: Projections for Ramses with anticipated unrestricted and restricted real and nominal interest rate (top row) and unanticipated restrictions on nominal interest rate (bottom row): 6-quarter restriction



projection to equal unity for quarters 0–5 corresponds to a very high and falling real instrument-rate projection. For Ramses, we have to extend the restrictions to 11 quarters, quarters 0–10, for the phenomenon to occur, as we can see in figure 8.6. Hence, inflation in Ramses appears to be less sensitive to the real interest rate in the Lindé model, so longer restrictions are needed for the phenomenon to occur.

In the bottom panel, figures 8.5 and 8.6 show the result of a projection with the Leeper-Zha method of modest interventions. There, positive unanticipated shocks (denoted by circles) are added to the estimated instrument rule to achieve the restriction on the nominal instrument rate. Comparing the bottom panel to the top right panel, we see that the impact on inflation, the output gap, and the real interest rate is smaller for the unanticipated shocks in the Leeper-Zha method than for the anticipated time-varying constants in our method.

Figure 8.6: Projections for Ramses with anticipated unrestricted and restricted real and nominal interest rate (top row) and unanticipated restrictions on nominal interest rate (bottom row): 11-quarter restriction



9. Conclusions

In describing the optimal policy choice, we have found it practical to describe it as the selection of an optimal policy projection of the instrument rate and the target variables from the set of feasible policy projections, the set of projections of the instrument rate and the target variables that satisfy the projection model of the economy. Thus, the optimal policy projection is the feasible projection that minimizes the intertemporal loss function that represents the objectives of monetary policy. Although this policy choice is formally equivalent to the choice of an optimal instrument rule, we find the above description of the optimal policy choice closer to the actual choice situation of a monetary-policy committee such as the Riksbank's Executive Board. Riksbank Board members see alternative policy projections to choose from, not alternative optimal instrument rules. Indeed, the optimal instrument rule implicitly takes into account all information that has an impact on the forecast of the target variables, including judgment, which makes it more or less infeasible to write it down.

From this point of view of the optimal policy choice, the set of feasible policy projections has an infinite number of elements. The methods to construct alternative policy projections discussed in this paper can then be seen as aiming to restrict the policy choice to a finite number of relevant alternative policy projections.

From the analysis in this paper, we conclude that, in a model without forward-looking variables, such as the empirical model of the U.S. economy by Rudebusch and Svensson [18], there is no difference between policy simulations with anticipated and unanticipated restrictions on the instrument-rate path. In a model with forward-looking variables, such as Ramses or the empirical New Keynesian model of the U.S. economy by Lindé [15], there is such a difference, and the impact of anticipated restrictions would generally be larger than that of unanticipated restrictions. In a model with forward-looking variables, exogenous restrictions on the instrument-rate path are consistent with a unique equilibrium, if there is a switch to a well-behaved policy rule in the future. For given restrictions on the instrument-rate path, the equilibrium depends on that policy rule.

Furthermore, our analysis shows that, if inflation is sufficiently sensitive to the real instrument rate, "unusual" equilibria may result from restrictions on the nominal instrument rate. Such cases have the property that a shift up of the real interest-rate path reduces inflation and inflation-expectations so much that the nominal interest-rate path (which by the Fisher equation equals the real interest-rate path plus the path of inflation expectations) shifts down. Then, a shift up of the nominal interest-rate path requires an equilibrium where the path of inflation and inflation expectations shifts up more and the real instrument-rate path shifts down. In the Rudebusch-Svensson model, which has no forward-looking variables, inflation is so sluggish and insensitive to changes in the real instrument rate that there are only small differences between restrictions on the nominal and real instrument rate. In the Lindé model, inflation is so sensitive to the real instrument rate that restrictions for 5–6 quarters or more on the nominal instrument rate results in unusual equilibria. In Ramses, unusual equilibrium seem to require restrictions for 11 quarters or more.

Because of the possibility of unusual equilibria for restrictions on nominal instrument rates, it is preferable to impose restrictions on the real rather than nominal instrument rate, since they lead to more robust and intuitive results. Also, it makes sense to impose restrictions on and consider shifts of the real rather than the nominal instrument-rate path, since the real interest rates rather than the nominal ones are what matter for real activity and for inflation. In general, given the possibility of unseal equilibria, it may in many cases actually be preferable to instead generate alternative instrument-rate projections as optimal policy projections that minimize an intertemporal loss function for alternative weights on the target variables.

Appendix

A. Unanticipated instrument-rule shocks: "Modest interventions" as in Leeper and Zha [14]

The method of "modest interventions" of Leeper and Zha [14] can be interpreted as generating central-bank projections that satisfy the restriction on the instrument rate by adding a sequence of additive shocks to the policy rule. These planned shocks are unanticipated by the private sector.

In order to illustrate the Leeper and Zha [14] method of modest interventions, we set $T = 0$, in which case

$$z_t = \eta_{t,t}$$

and the deviation is a simple zero-mean i.i.d. disturbance. We can then write the projection model as perceived by the private sector as

$$\begin{bmatrix} \tilde{X}_{t+\tau+1,t} \\ \tilde{H}\tilde{x}_{t+\tau+1,t} \end{bmatrix} = \tilde{A} \begin{bmatrix} \tilde{X}_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix} \tag{A.1}$$

for $\tau \geq 0$, where

$$\tilde{X}_t \equiv \begin{bmatrix} X_t \\ z_t \end{bmatrix}, \quad \tilde{x}_t \equiv \begin{bmatrix} x_t \\ i_t \end{bmatrix}, \quad \tilde{H} \equiv \begin{bmatrix} H & 0 \\ G_x & G_i \end{bmatrix},$$

$$\tilde{A} \equiv \begin{bmatrix} A_{11} & 0_{n_X \times 1} & A_{12} & B_1 \\ 0_{1 \times n_X} & 0_{1 \times 1} & 0_{1 \times n_x} & 0_{1 \times 1} \\ A_{21} & 0_{n_x \times 1} & A_{22} & B_2 \\ f_X & 1 & f_x & f_i \end{bmatrix}.$$

The solution to this system can be written

$$\begin{aligned} \begin{bmatrix} X^p_{t+\tau,t} \\ 0 \end{bmatrix} &= M^\tau \tilde{X}_{t,t}, \\ \tilde{x}^p_{t+\tau,t} &\equiv \begin{bmatrix} x^p_{t+\tau,t} \\ i^p_{t+\tau,t} \end{bmatrix} = F \begin{bmatrix} X^p_{t+\tau,t} \\ 0 \end{bmatrix} = \begin{bmatrix} F_x \\ F_i \end{bmatrix} M^\tau \tilde{X}_{t,t} \end{aligned}$$

for $\tau \geq 0$, where the superscript p denotes that this is the projection believed by the private sector in period t.

Let us demonstrate the method of modest interventions only for the restriction (7.1). The central bank plans to satisfy this restriction by a sequence of shocks $\{\tilde{\eta}_{t+\tau,t}\}_{\tau=0}^{T}$ that are unanticipated by the private sector. These shocks are chosen such that $\tilde{\eta}_{t,t}$ satisfies

$$i_{t,t} = F_i \begin{bmatrix} X_{t,t} \\ \tilde{\eta}_{t,t} \end{bmatrix} = \bar{\imath}_{t,t}.$$

Then the projection of the current forward-looking variables is given by

$$x_{t,t} = F_x \begin{bmatrix} X_{t,t} \\ \tilde{\eta}_{t,t} \end{bmatrix}.$$

For $\tau = 1, ..., T$, the projection of the predetermined variables is then given by

$$\begin{bmatrix} X_{t+\tau,t} \\ 0 \end{bmatrix} = M \begin{bmatrix} X_{t+\tau-1,t} \\ \tilde{\eta}_{t+\tau-1,t} \end{bmatrix},$$

the shock $\tilde{\eta}_{t+\tau,t}$ is chosen to satisfy

$$i_{t+\tau,t} = F_i \begin{bmatrix} X_{t+\tau,t} \\ \tilde{\eta}_{t+\tau,t} \end{bmatrix} = \bar{\imath}_{t+\tau,t},$$

and the projection of the forward-looking variables is given by

$$x_{t+\tau,t} = F_x \begin{bmatrix} X_{t+\tau,t} \\ \tilde{\eta}_{t+\tau,t} \end{bmatrix}.$$

There are some conceptual difficulties in a central bank announcing such an instrument-rate path and projection to the private sector. The projection is only relevant if the private sector does not believe that the central bank will actually implement the path but instead follow the instrument rule with zero expected instrument-rule shocks. The method of modest interventions is instead perhaps more appropriate for secret policy simulations and plans that are not announced to the private sector.

B. Projections with an arbitrary time-varying instrument rule

As an alternative method to construct projections with arbitrary time-varying instrument-rate paths, consider projections in period t with an arbitrary *time-varying* general policy rule, hence including time-varying targeting and instrument rules. A time-varying instrument rule gives considerable flexibility in constructing projections. For instance, when one of the elements of X_t is unity, which is a practical way to handle non-zero intercepts in the model equations, the instrument rule can be chosen such that all response coefficients are zero except for the coefficient that corresponds to this unitary element of X_t. Making that coefficient time varying is then a simple way of implementing an arbitrary exogenous path for the nominal or real instrument rate.

More precisely, let (G_τ, f_τ) for $\tau \geq 0$ denote the general time-varying policy rule,

$$G_\tau \tilde{x}_{t+\tau+1,t} = f_\tau \begin{bmatrix} X_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix}, \tag{B.1}$$

where

$$G_\tau = [G_{x\tau} \quad G_{i\tau}], \quad f_\tau \equiv [f_{X\tau} \quad f_{x\tau} \quad f_{i\tau}]$$

are partitioned conformably with x_t and i_t, and with X_t, x_t, and i_t, respectively. In particular, for $\tau \geq T$, let the policy rule satisfy

$$G_\tau = G_T, \quad f_\tau \equiv f_T.$$

That is, from T periods ahead the policy rule is no longer time-varying but constant. Restricting the policy rule rule to be constant beyond some finite horizon is a practical way to ensure determinacy of the projection (the saddlepoint property).

Assume that the constant policy rule (G_T, f_T) makes the system satisfy the saddlepoint property. Then we know from the previous section that there exist matrices M_T and F_T such that the resulting equilibrium projection satisfies

$$X_{t+\tau+1,t} = M_T X_{t+\tau,t},$$

$$\tilde{x}_{t+\tau,t} \equiv \begin{bmatrix} x_{t+\tau,t} \\ i_{t+\tau,t} \end{bmatrix} = F_T X_{t+\tau,t} \equiv \begin{bmatrix} F_{xT} \\ F_{iT} \end{bmatrix} X_{t+\tau,t} \tag{B.2}$$

for $\tau \geq T$.

Now, consider $\tau = T-1$. First, we define

$$\tilde{H}_\tau \equiv \begin{bmatrix} H & 0 \\ G_{x\tau} & G_{i\tau} \end{bmatrix}, \quad \tilde{A}_\tau \equiv \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ f_{X\tau} & f_{x\tau} & f_{i\tau} \end{bmatrix} \equiv \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21\tau} & \tilde{A}_{22\tau} \end{bmatrix},$$

where we partition $\tilde{A}_\tau$ conformably with X_t and $\tilde{x}_t$ and observe that $\tilde{A}_{11} \equiv A_{11}$ and $\tilde{A}_{12} \equiv [A_{12} \quad B_1]$ do not depend on τ. We then have

$$X_{t+T,t} = \tilde{A}_{11} X_{t+T-1,t} + \tilde{A}_{12} \tilde{x}_{t+T-1,t}, \tag{B.3}$$

$$\begin{aligned} \tilde{H}_{T-1} \tilde{x}_{t+T,t} &= \tilde{H}_{T-1} F_T X_{t+T,t} \\ &= \tilde{H}_{T-1} F_T (\tilde{A}_{11} X_{t+T-1,t} + \tilde{A}_{12} \tilde{x}_{t+T-1,t}) \\ &= \tilde{A}_{21,T-1} X_{t+T-1,t} + \tilde{A}_{22,T-1} \tilde{x}_{t+T-1,t}, \end{aligned} \tag{B.4}$$

where in the latter equation we use (B.2) in the first equality and (B.3) in the second. Equation (B.4) provides an equation for $x_{t+T-1,t}$ for given $X_{t+T-1,t}$, F_T, $\tilde{H}_{T-1}$, and $\tilde{A}_{T-1}$. Solving (B.4) for $x_{t+T-1,t}$ gives

$$\tilde{x}_{t+T-1,t} \equiv \begin{bmatrix} x_{t+T-1,t} \\ i_{t+T-1,t} \end{bmatrix} = F_{T-1} X_{t+T-1,t} \equiv \begin{bmatrix} F_{x,T-1} \\ F_{i,T-1} \end{bmatrix} X_{t+T-1,t},$$

where[16]

$$F_{T-1} \equiv (\tilde{A}_{22,T-1} - \tilde{H}_{T-1} F_T \tilde{A}_{12})^{-1} (H_{T-1} F_T \tilde{A}_{11} - \tilde{A}_{21,T-1}). \tag{B.5}$$

Furthermore, we get

$$X_{t+T,t} = M_{T-1} X_{t+T-1,t},$$

where

$$M_{T-1} \equiv \tilde{A}_{11} + \tilde{A}_{12} F_{T-1}. \tag{B.6}$$

Thus, equations (B.5) and (B.6) imply a mapping from (F_T, M_T) to (F_{T-1}, M_{T-1}). It is now obvious that equations (B.5) and (B.6) and their analogues for $\tau = T-2, T-3, ..., 0$ allow us to recursively construct F_τ and M_τ for $\tau = T-1, ..., 0$, such that the equilibrium projection for a time-varying policy rule can be written

$$X_{t+\tau+1,t} = M_\tau X_{t+\tau,t},$$

$$\tilde{x}_{t+\tau,t} \equiv \begin{bmatrix} x_{t+\tau,t} \\ i_{t+\tau,t} \end{bmatrix} = F_\tau X_{t+\tau,t} \equiv \begin{bmatrix} F_{x\tau} \\ F_{i\tau} \end{bmatrix} X_{t+\tau,t},$$

$$Y_{t+\tau,t} = D \begin{bmatrix} X_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix}$$

for $\tau \geq 0$, where $M_\tau = M_T$ and $F_\tau = F_T$ for $\tau \geq T$.

For arbitrary (G_τ, f_τ) for $\tau = 0, 1, ..., T$, where (G_T, f_T) apply for $\tau \geq T$, there is a possibility that there is a sizeable jump in the projection of the instrument rate and other variables between $\tau = T-1$ and $\tau = T$. In many cases alternative instrument-rate paths will be more meaningful and more relevant alternatives if they do not have such discontinuities. This may require some experimentation with the sequence of (G_τ, f_τ), the constant future policy rule (G_T, f_T), and the horizon T. Note again that complete control of the instrument-rate path up to $\tau = T-1$ can be achieved when $G_\tau = 0$, and $f_\tau \equiv [f_{X\tau} \quad f_{x\tau} \quad f_{i\tau}]$ for $\tau = 0, 1, ..., T-1$ satisfies $f_{i\tau} \equiv -1$ and has otherwise zero response coefficients except the element of the vector of predetermined variables that is set equal to unity.

[16] We assume that the matrix $\tilde{A}_{22,T-1} - H_{T-1} F_T \tilde{A}_{12}$ is nonsingular, which is a condition that the nonpredetermined variables in period $t+T-1$ ($\tilde{x}_{t+T-1}$) are determined by the predetermined variables in period $t+T-1$ (X_{t+T-1}) and the expectations $\mathrm{E}_{t+T-1} \tilde{x}_{t+T}$ in period $t+T-1$ of the nonpredetermined variables in period $t+T$ ($\tilde{x}_{t+T}$). This condition is satisfied by an economically meaningful model and an economically meaningful policy rule.

B.1. Real instrument rate

We can express any time-varying policy rule (G_τ, f_τ) in terms of the real instrument rate by using (7.3) and (7.7) to substitute for i_t in the policy rule,

$$\begin{aligned} G_\tau \tilde{x}_{t+\tau+1,t} &= f_{X\tau} X_{t+\tau,t} + f_{x\tau} x_{t+\tau,t} + f_{i\tau} i_{t+\tau,t} \\ &= f_{X\tau} X_{t+\tau,t} + f_{x\tau} x_{t+\tau,t} + f_{i\tau} r_{t+\tau,t} + f_{i\tau} \Phi \begin{bmatrix} X_{t+\tau\, t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix} + f_{i\tau} \varphi \tilde{x}_{t+\tau+1,t}. \end{aligned}$$

We can rewrite this policy rule as as

$$G^r_\tau \tilde{x}_{t+\tau+1,t} = f^r_{X\tau} X_{t+\tau,t} + f^r_{\tilde{x}\tau} \tilde{x}_{t+\tau,t} + f^r_{r\tau} r_{t+\tau,t},$$

where

$$\begin{aligned} G^r_\tau &\equiv G_\tau - f_{i\tau}\varphi, \\ f^r_{X\tau} &\equiv f_{X\tau} + f_{i\tau}\Phi_X, \\ f^r_{\tilde{x}} &\equiv \begin{bmatrix} f_{x\tau} \\ 0 \end{bmatrix} + f_{i\tau}\Phi_{\tilde{x}}, \\ f^r_{r\tau} &\equiv f_{i\tau}, \end{aligned}$$

where $\Phi \equiv [\Phi_X \;\; \Phi_{\tilde{x}}]$ is partitioned conformably with X_t and $\tilde{x}_t$. This gives us a policy rule in terms of the real instrument rate.

The initial projection $(\bar{X}^t, \bar{x}^t, \bar{\imath}^t, \bar{Y}^t)$ will imply a projection $\bar{r}^t = \{\bar{r}_{t+\tau,t}\}_{\tau=0}^\infty$ for the real instrument rate, which will satisfy $\bar{r}_{t+\tau,t} \equiv \bar{\imath}_{t+\tau,t} - \bar{\pi}_{t+1+\tau,1}$ for $\tau \geq 0$, where $\bar{\pi}^t \equiv \{\bar{\pi}_{t+\tau,t}\}_{\tau=0}^\infty$ is the projection for inflation. We can now choose the time-varying policy rule such that, for $\tau = 0, 1, ..., T-1$, $G^r_\tau = 0$, $f^r_{\tilde{x}\tau} = 0$, $f^r_{r\tau} = -1$, $f^r_{Xj\tau} = \bar{r}_{t+\tau,t}$ for $j = 0$, and $f^r_{Xj\tau} = 0$ for $j = 1, ..., n_X - 1$. For $\tau \geq T$, we choose the policy rule to equal the initial policy rule. That is, for $\tau = 0, ..., T-1$, this policy rule makes the real instrument rate exogenous and equal to the initial projection of the real instrument rate. In order to express this policy rule in terms of the nominal instrument rate, we can use (7.3) and (7.7) to substitute for $r_{t+\tau,t}$,

$$\begin{aligned} 0 &= f^r_{X0\tau} - r_{t+\tau,t} \\ &= f^r_{X0\tau} - i_{t+\tau,t} + \varphi \tilde{x}_{t+\tau+1,t} + \Phi \begin{bmatrix} X_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix}, \end{aligned}$$

which we can write as

$$-\varphi \tilde{x}_{t+\tau+1,t} = f^r_{X0\tau} + \Phi_X X_{t+\tau,t} + \Phi_x x_{t+\tau,t} + (\Phi_i - 1) i_{t+\tau,t}, \tag{B.7}$$

which is on the form (B.1).

It should now be clear that the projection for the time-varying policy rule (B.7) will be equal to the initial projection. Hence, we can generate the initial projection with an exogenous path for the real instrument rate. By varying the terms $f^r_{X0\tau}$ for $\tau = 0, 1, ..., T-1$, we can then generate projections for alternative exogenous real instrument rate paths.

C. The Rudebusch-Svensson model: An empirical backward-looking model

The two equations of the model of Rudebusch and Svensson [18] are

$$\pi_{t+1} = \alpha_{\pi 1}\pi_t + \alpha_{\pi 2}\pi_{t-1} + \alpha_{\pi 3}\pi_{t-2} + \alpha_{\pi 4}\pi_{t-3} + \alpha_y y_t + z_{\pi,t+1} \tag{C.1}$$

$$y_{t+1} = \beta_{y1} y_t + \beta_{y2} y_{t-1} - \beta_r \left(\frac{1}{4}\Sigma^3_{j=0} i_{t-j} - \frac{1}{4}\Sigma^3_{j=0}\pi_{t-j} \right) + z_{y,t+1}, \tag{C.2}$$

where π_t is quarterly inflation in the GDP chain-weighted price index (P_t) in percentage points at an annual rate, i.e., $400(\ln P_t - \ln P_{t-1})$; i_t is the quarterly average federal funds rate in percentage points at an annual rate; y_t is the relative gap between actual real GDP (Q_t) and potential GDP (Q^*_t) in percentage points, i.e., $100(Q_t - Q^*_t)/Q^*_t$. These five variables were demeaned prior to estimation, so no constants appear in the equations.

The estimated parameters, using the sample period 1961:1 to 1996:2, are shown in table C.1.

Table C.1

$\alpha_{\pi 1}$	$\alpha_{\pi 2}$	$\alpha_{\pi 3}$	$\alpha_{\pi 4}$	α_y	β_{y1}	β_{y2}	β_r
0.70	−0.10	0.28	0.12	0.14	1.16	−0.25	0.10
(0.08)	(0.10)	(0.10)	(0.08)	(0.03)	(0.08)	(0.08)	(0.03)

The hypothesis that the sum of the lag coefficients of inflation equals one has a p-value of .16, so this restriction was imposed in the estimation.

The state-space form can be written

$$\begin{bmatrix} \pi_{t+1} \\ \pi_t \\ \pi_{t-1} \\ \pi_{t-2} \\ y_{t+1} \\ y_t \\ i_t \\ i_{t-1} \\ i_{t-2} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^4 \alpha_{\pi j} e_j + \alpha_y e_5 \\ e_1 \\ e_2 \\ e_3 \\ \beta_r e_{1:4} + \beta_{y1} e_5 + \beta_{y2} e_6 - \beta_r e_{7:9} \\ e_5 \\ e_0 \\ e_7 \\ e_8 \end{bmatrix} \begin{bmatrix} \pi_t \\ \pi_{t-1} \\ \pi_{t-2} \\ \pi_{t-3} \\ y_t \\ y_{t-1} \\ i_{t-1} \\ i_{t-2} \\ i_{t-3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\beta_r}{4} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} i_t + \begin{bmatrix} z_{\pi,t+1} \\ 0 \\ 0 \\ 0 \\ z_{y,t+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where e_j ($j = 0, 1, ..., 9$) denotes a $1{\times}9$ row vector, for $j = 0$ with all elements equal to zero, for $j = 1, ..., 9$ with element j equal to unity and all other elements equal to zero; and where $e_{j:k}$

$(j < k)$ denotes a 1×9 row vector with elements $j, j+1, ..., k$ equal to $\frac{1}{4}$ and all other elements equal to zero. The predetermined variables are π_t, π_{t-1}, π_{t-2}, π_{t-3}, y_t, y_{t-1}, i_{t-1}, i_{t-2}, i_{t-2}, and i_{t-3}. There are no forward-looking variables.

For a loss function (8.3) with $\delta = 1$, $\lambda = 1$, and $\nu = 0.2$, and the case where z_t is an i.i.d. zero-mean shock; the optimal instrument rule is (the coefficients are rounded to two decimal points),

$$i_t = 1.22\,\pi_t + 0.43\,\pi_{t-1} + 0.53\,\pi_{t-2} + 0.18\,\pi_{t-3} + 1.93\,y_t - 0.49\,y_{t-1} + 0.36\,i_{t-1} - 0.09\,i_{t-2} - 0.05\,i_{t-3}.$$

D. The Lindé model: An empirical New Keynesian model

An empirical New Keynesian model estimated by Lindé [15] is

$$\begin{aligned} \pi_t &= \omega_f \pi_{t+1|t} + (1-\omega_f)\pi_{t-1} + \gamma y_t + \varepsilon_{\pi t}, \\ y_t &= \beta_f y_{t+1|t} + (1-\beta_f)(\beta_{y1} y_{t-1} + \beta_{y2} y_{t-2} + \beta_{y3} y_{t-3} + \beta_{y4} y_{t-4}) - \beta_r (i_t - \pi_{t+1|t}) + \varepsilon_{yt}, \end{aligned}$$

where the restriction $\sum_{j=1}^{4} \beta_{yj} = 1$ is imposed and $\varepsilon_t \equiv (\varepsilon_{\pi t}, \varepsilon_{yt})'$ is an i.i.d. shock with mean zero. The estimated coefficients are (Table 6a in Lindé [15], non-farm business output) are shown in table D.1.

Table D.1

ω_f	γ	β_f	β_r	β_{y1}	β_{y2}	β_{y3}
0.457	0.048	0.425	0.156	1.310	− 0.229	− 0.011
(0.065)	(0.007)	(0.027)	(0.016)	(0.174)	(0.279)	(0.037)

For simplicity, we set $\beta_{y1} = 1$, $\beta_{y2} = \beta_{y3} = \beta_{y4} = 0$. Then the state-space form can be written as

$$\begin{bmatrix} \varepsilon_{\pi,t+1} \\ \varepsilon_{y,t+1} \\ \pi_t \\ y_t \\ i_t \\ \omega_f \pi_{t+1|t} \\ \beta_r \pi_{t+1|t} + \beta_f y_{t+1|t} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -(1-\omega_f) & 0 & 0 & 1 & -\gamma \\ 0 & -1 & 0 & -(1-\beta_f) & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{\pi t} \\ \varepsilon_{yt} \\ \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \\ \pi_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \beta_r \end{bmatrix} i_t + \begin{bmatrix} \varepsilon_{\pi,t+1} \\ \varepsilon_{y,t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The predetermined variables are $\varepsilon_{\pi t}$, ε_{yt}, π_{t-1}, y_{t-1}, and i_{t-1}, and the forward-looking variables are π_t and y_t.

For a loss function (8.3) with $\delta = 1$, $\lambda_y = 1$, and $\lambda_{\Delta i} = 0.2$, and the case where ε_t is an i.i.d. zero-mean shock; the optimal instrument rule is (the coefficients are rounded to two decimal points),

$$i_t = 1.06\,\varepsilon_{\pi t} + 1.38\,\varepsilon_{yt} + 0.58\,\pi_{t-1} + 0.78\,y_{t-1} + 0.40\;i_{t-1} + 0.02\,\Xi_{\pi,t-1,t-1} + 0.20\,\Xi_{y,t-1,t-1},$$

where $\Xi_{\pi,t-1,t-1}$ and $\Xi_{y,t-1,t-1}$ are the Lagrange multipliers for the two equations for the forward-looking variables in the decision problem in period $t-1$. The difference equation (6.5) for the Lagrange multipliers is

$$\begin{bmatrix} \Xi_{\pi t} \\ \Xi_{yt} \end{bmatrix} = \begin{bmatrix} 10.20 & 0.74 & 5.54 & 0.43 & -0.21 \\ 0.74 & 1.48 & 0.40 & 0.85 & -0.28 \end{bmatrix} \begin{bmatrix} \varepsilon_{\pi t} \\ \varepsilon_{yt} \\ \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} 0.72 & 0.16 \\ 0.03 & 0.38 \end{bmatrix} \begin{bmatrix} \Xi_{\pi,t-1} \\ \Xi_{y,t-1} \end{bmatrix}.$$

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