Forecasting uncertainty of output gap estimates*

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Abstract

This paper investigates the uncertainty around empirical output gap forecasts in a state-space model framework. Various methods to obtain mean squared forecast errors that take into account estimation uncertainty in alternative ways are compared in a Monte Carlo experiment. A bootstrap method that performs best in the simulations is applied to estimate the uncertainty surrounding the output gap in the Euro area.

1 Introduction

Output gap measures play an important role both in economic theory and policy. In recent theoretical models with imperfect price flexibility, the movement of the output gap, defined as output minus trend or potential output, leads to inflation or deflation.[1] Therefore, output gap measures are important empirical indicators for economic policy, especially monetary policy. Up to now, a lot of studies already exist that measure output gaps in the Euro area.[2] However, most of these measures are solely computed ex-post. In this paper, ex-ante forecasts of the output gap are calculated in addition to the existing literature. To implement the forecasts empirically, a state space framework is chosen to decompose output into trend and the output gap. To highlight the uncertainty around the output gap forecasts, we construct ex-ante prediction intervals using a bootstrap method recently proposed by Pfeffermann/Tiller (2003). The simulated forecast intervals based on this method consider the uncertainty that is due to estimation of the model parameters. The usefulness and small sample accuracy of the PT method is investigated in a Monte Carlo experiment. Moreover, output gap forecasts and corresponding forecast intervals are computed for the Euro area. For this purpose, two alternative bivariate state space models for output and inflation following Kuttner (1994) as well as output and capacity utilization following Rünstler (2002) are estimated.

The paper proceeds as follows: In section 2, forecasting in a state space framework is introduced, and the Pfeffermann/Tiller (2003) method for the bootstrap computation of forecast mean squared errors (MSE) is described. Section 3 provides some Monte Carlo evidence on the small sample properties. In section 4, the output gap is estimated and forecast for Euro area data. Section 5 concludes.

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[1]See Clarida et al. (2000).

[2]See, for example, Gerlach/Smets (1999), Rünstler (2002), Camba-Méndez/Palenzuela (2003).

2 Forecasting with state space models

The following section describes the state space model and how we can obtain forecasts and measures of forecast uncertainty. The notation follows Hamilton (1994), chapter 13.

State space model A non time-varying state space model consists of the two vector equations

$$\boldsymbol{\xi}_{t+1} = \mathsf{F}\boldsymbol{\xi}_t + \mathsf{v}_{t+1}, \tag{1}$$

$$\mathsf{y}_t = \mathsf{H}'\boldsymbol{\xi}_t + \mathsf{w}_t, \tag{2}$$

where the first equation is the state equation, which is an autoregressive dynamic equation for the $(r \times 1)$ state vector $\boldsymbol{\xi}_t$. The disturbance v_t has covariance Q and is not serially correlated. F is the transition matrix. The observation equation (2) links the $(n \times 1)$ observable variables vector y_t to the state through the $(n \times r)$ matrix H'. The disturbance w_t has covariance R and is not serially correlated, too. Up to now, it is assumed that the parameter matrices are known. The two types of disturbances are assumed to be uncorrelated at all leads and lags, $E(\mathsf{v}_t\mathsf{w}'_\tau) = 0$ for all t and τ. The state vector is typically not observed. However, for given distributions of the disturbances, the Kalman filter can be applied to obtain optimal predictions of the state. More precisely, the Kalman filter can be motivated as an algorithm for calculating linear least squares forecasts of the state vector for period $t+1$ on the basis of information available in period t,

$$\widehat{\boldsymbol{\xi}}_{t+1|t} \equiv \widehat{E}\left(\boldsymbol{\xi}_{t+1}|\mathcal{I}_t\right), \tag{3}$$

where $\mathcal{I}_T$ is observable time series information available in period t. The Kalman filter calculates these forecasts recursively, starting from an initial guess up to the end of the sample $t = 1, 2, \ldots, T$. Associated with the state forecasts, a mean squared error, henceforth MSE, matrix can be computed to assess the filter uncertainty around the state:

$$\mathsf{P}_{t+1|t} = E\left[(\boldsymbol{\xi}_{t+1} - \widehat{\boldsymbol{\xi}}_{t+1|t})(\boldsymbol{\xi}_{t+1} - \widehat{\boldsymbol{\xi}}_{t+1|t})'\right]. \tag{4}$$

During the Kalman filter recursions, the observed variables y_t can be used to improve and update the estimates of the state. This leads to an updated filtered state $\widehat{\boldsymbol{\xi}}_{t|t}$ and its MSE $\mathsf{P}_{t|t}$.[3] Since the Kalman filtering allows for optimal estimation of unobserved states, such a model is a useful tool to model typically unobserved potential output and output gaps.

Forecasting with known parameters The forecast of the state h periods ahead is given by a recursive solution of the state equation

$$\widehat{\boldsymbol{\xi}}_{T+h|T} = \mathsf{F}^h\widehat{\boldsymbol{\xi}}_{T|T}, \tag{5}$$

where $\widehat{\boldsymbol{\xi}}_{T+h|T}$ denotes conditional expectation according to $\widehat{\boldsymbol{\xi}}_{T+h|T} = \widehat{E}\left(\boldsymbol{\xi}_{T+h}|\mathcal{I}_T\right)$, and $\mathcal{I}_T$ is the observable time series information available in period T. Shock terms from the state equation (1) vanish, because the expectation of shocks out-of-sample given information in period T is equal to zero. Hence, the transition matrix F and the updated filtered state at the end of the sample $\widehat{\boldsymbol{\xi}}_{T|T}$ are needed to forecast the state h periods ahead. From the state equation, we can

[3] For details, see Hamilton (1994), pp. 379 or Harvey (1989), pp. 105.

obtain the true state $\boldsymbol{\xi}_{T+h} = \mathsf{F}^h \boldsymbol{\xi}_T + \mathsf{F}^{h-1}\mathsf{v}_{T+1} + \ldots + \mathsf{F}^1\mathsf{v}_{T+h-1} + \mathsf{v}_{T+h}$. The forecast error is then given by $\boldsymbol{\xi}_{T+h} - \hat{\boldsymbol{\xi}}_{T+h|T}$, and we can obtain the the forecast MSE according to

$$\mathsf{P}_{T+h|T} = \mathsf{F}^h \mathsf{P}_{T|T}(\mathsf{F}')^h + \mathsf{F}^{h-1}\mathsf{Q}(\mathsf{F}')^{h-1} + \mathsf{F}^{h-2}\mathsf{Q}(\mathsf{F}')^{h-2} + \ldots + \mathsf{F}\mathsf{Q}\mathsf{F}' + \mathsf{Q}, \qquad (6)$$

which depends only on the system matrices F, Q, and the end-of-sample MSE $\mathsf{P}_{T|T}$. A forecast interval for an element i out of the state vector is then given by

$$\hat{\xi}_{i,T+h|T} \pm 1.96\sqrt{\mathsf{P}_{i,i,T+h|T}}, \qquad (7)$$

with 95 % coverage. $\mathsf{P}_{i,i,T+h|T}$ denotes the (i,i) element of the MSE matrix.[4]

Forecasting with unknown parameters The typical procedure to obtain forecast intervals is to estimate the hyperparameters of the state space model, and calculate the MSE with these estimated parameters. This neglects the sampling uncertainty about the true values of the model's parameters. Denote the filtered state forecast which is obtained by inserting the true parameters by $\hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the vector of true hyperparameters. $\mathsf{P}_{T+h|T}(\boldsymbol{\theta})$ is the corresponding MSE given by

$$\mathsf{P}_{T+h|T}(\boldsymbol{\theta}) = E\left[(\boldsymbol{\xi}_{T+h} - \hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta}))(\boldsymbol{\xi}_{T+h} - \hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta}))'\right]. \qquad (8)$$

Up to now, it was assumed that the ML estimator of the parameters, $\hat{\boldsymbol{\theta}}$, is equal to the true $\boldsymbol{\theta}$. Denote the state filter forecast which is obtained by using the ML estimators by $\hat{\boldsymbol{\xi}}_{T+h|T}(\hat{\boldsymbol{\theta}})$, where $\hat{\boldsymbol{\theta}}$ is the vector of estimated hyperparameters. $\mathsf{P}_{T+h|T}(\hat{\boldsymbol{\theta}})$ is the corresponding MSE given by

$$\begin{aligned}
\mathsf{P}_{T+h|T}(\hat{\boldsymbol{\theta}}) &= E\left[(\boldsymbol{\xi}_{T+h} - \hat{\boldsymbol{\xi}}_{T+h|T}(\hat{\boldsymbol{\theta}}))(\boldsymbol{\xi}_{T+h} - \hat{\boldsymbol{\xi}}_{T+h|T}(\hat{\boldsymbol{\theta}}))'\right] && (9)\\
&= E\left[(\boldsymbol{\xi}_{T+h} - \hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta}))(\boldsymbol{\xi}_{T+h} - \hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta}))'\right] \\
&\quad + E\left[(\hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta}) - \hat{\boldsymbol{\xi}}_{T+h|T}(\hat{\boldsymbol{\theta}}))(\hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta}) - \hat{\boldsymbol{\xi}}_{T+h|T}(\hat{\boldsymbol{\theta}}))'\right] && (10)\\
&= \mathsf{P}_{T+h|T}(\boldsymbol{\theta}) + E\left[(\hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta}) - \hat{\boldsymbol{\xi}}_{T+h|T}(\hat{\boldsymbol{\theta}}))(\hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta}) - \hat{\boldsymbol{\xi}}_{T+h|T}(\hat{\boldsymbol{\theta}}))'\right]. && (11)
\end{aligned}$$

From the first to the second equaltiy, $\pm\hat{\boldsymbol{\xi}}_{T+h|T}(\boldsymbol{\theta})$ was added in both brackets and then multiplied out. The equality (10) decomposes the MSE into $\mathsf{P}_{T+h|T}(\boldsymbol{\theta})$ from (8), which is a measure of filter uncertainty when the true vector is known, and a measure of estimation uncertainty, which shows the volatility between filtered states when the true and the estimated hyperparameters are used.[5] Therefore, when the true parameters are not known, assuming the equality $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ leads to an underestimation of the true variability of the state vector and the MSE, because the second term on the right hand side of (11) is neglected. In the literature exist various methods to take into account this estimation uncertainty. In the following, these measures will be compared with the 'naive' MSE, denoted as $\mathsf{P}^{\mathrm{N}}_{T+h|T}(\hat{\boldsymbol{\theta}})$, which is obtained by using the estimated parameters as the true parameters in the Kalman filter recursions and the forecasting equation (6).

[4] See Harvey (1989), p. 223.
[5] See Pfeffermann/Tiller (2003), p. 6 and Hamilton (1994), p. 398.

Ansley/Kohn (1986) The method employed by Ansley/Kohn (1986), in the following AK, aims at approximating the second term on the right hand side of (10). The delta method is employed to obtain the variance of the state which is due to parameter uncertainty. Filter uncertainty is simply approximated by the naive MSE $\mathsf{P}^{\mathrm{N}}_{T+h|T}(\hat{\boldsymbol{\theta}})$. The AK MSE is given by

$$\mathsf{P}^{\mathrm{AK}}_{T+h|T}(\hat{\boldsymbol{\theta}}) = \mathsf{P}^{\mathrm{N}}_{T+h|T}(\hat{\boldsymbol{\theta}}) + \left(\left.\frac{\partial \boldsymbol{\xi}_{T+h}}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}\right)' \left(\left.\frac{1}{T}\mathsf{I}^{-1}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}\right) \left(\left.\frac{\partial \boldsymbol{\xi}_{T+h}}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}\right), \tag{12}$$

where $\left.\frac{1}{T}\mathsf{I}^{-1}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$ denotes the inverse of the information matrix divided by T evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$. This matrix is a by-product of the ML estimation of the state space model and is equal to the estimated variance of the hyperparameters. The partial derivatives $\left.\partial \boldsymbol{\xi}_{T+h}/\partial \boldsymbol{\theta}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$ can be obtained by running the Kalman filter for small deviations from $\hat{\boldsymbol{\theta}}$.[6] A drawback of this method is that the information matrix may become unstable when the number of parameter is large.[7] Moreover, Quenneville/Singh (2000) have shown that approximating the filter uncertainty by the naive MSE yields a biased estimate.[8]

Pfeffermann/Tiller (2003) The MSE approximation by Pfeffermann/Tiller (2003), in the following PT, is a parametric bootstrap method which does not depend on the information matrix and removes the bias in the AK approximation. The PT method proceeds in the following steps:

1. In the first step, the estimated state space model with parameters $\hat{\boldsymbol{\theta}}$ is used to replicate the observables in (2) B times, say $\mathsf{y}_{b,t}$, for $b = 1, \ldots, B$ with a B large number. For this purpose, Pfeffermann/Tiller (2003) employ parametric resampling from the disturbances v_{t+1} and w_t, where the normality assumption is assumed to hold: $\mathsf{v}_t \sim N(\mathsf{0}, \mathsf{Q})$ and $\mathsf{w}_t \sim N(\mathsf{0}, \mathsf{R})$. The dynamic structure of the state space model (1) and (2) then gives the artificial data sets $\mathsf{y}_{b,t}$.[9]

2. The new data $\mathsf{y}_{b,t}$ is then used to reestimate the state space model to get $\hat{\boldsymbol{\theta}}_b$.

3. The Kalman filter can now be applied to each new data set and the new parameters to provide a forecast $\hat{\boldsymbol{\xi}}_{b,T+h|T}(\hat{\boldsymbol{\theta}}_b)$. An additional Kalman filter run with the new data, and the baseline parameter estimate $\hat{\boldsymbol{\theta}}$ gives the forecast $\hat{\boldsymbol{\xi}}_{b,T+h|T}(\hat{\boldsymbol{\theta}})$. The distribution of the deviations of these two states $\hat{\boldsymbol{\xi}}_{b,T+h|T}(\hat{\boldsymbol{\theta}}_b) - \hat{\boldsymbol{\xi}}_{b,T+h|T}(\hat{\boldsymbol{\theta}})$ is an approximation to the estimation uncertainty when B is large.

4. As an approximation to filter uncertainty, for each bootstrap data set, the naive MSE $\mathsf{P}^{\mathrm{N}}_{b,T+h|T}(\hat{\boldsymbol{\theta}}_b)$ is computed.

By averaging over the the terms for estimation and filter uncertainty, we can obtain the MSE matrix proposed by PT

$$\begin{aligned}\mathsf{P}^{\mathrm{PT}}_{T+h|T}(\hat{\boldsymbol{\theta}}) &= \frac{1}{B}\sum_{b=1}^{B}(\hat{\boldsymbol{\xi}}_{b,T+h|T}(\hat{\boldsymbol{\theta}}_b) - \hat{\boldsymbol{\xi}}_{b,T+h|T}(\hat{\boldsymbol{\theta}}))(\hat{\boldsymbol{\xi}}_{b,T+h|T}(\hat{\boldsymbol{\theta}}_b) - \hat{\boldsymbol{\xi}}_{b,T+h|T}(\hat{\boldsymbol{\theta}}))' \\ &\quad + \left(2\mathsf{P}^{\mathrm{N}}_{T+h|T}(\hat{\boldsymbol{\theta}}) - \frac{1}{B}\sum_{b=1}^{B}\mathsf{P}^{\mathrm{N}}_{b,T+h|T}(\hat{\boldsymbol{\theta}}_b)\right).\end{aligned} \tag{13}$$

[6] See Harvey (1989), p. 149 and pp. 142.
[7] See Pfeffermann/Tiller (2003), p. 11.
[8] Quenneville/Singh (2000), p. 224.
[9] A non-parametric alternative to this step is proposed by Stoffer/Wall (1991).

The first term on the right hand side is the bootstrap counterpart of the estimation uncertainty measure in (10). The term in brackets on the right hand side is a bias corrected bootstrap estimate of the filter uncertainty. Convergence results for the the MSE can be found in Pfeffermann/Tiller (2003). It must be kept in mind, that the PT method can be very burdensome from a computationally point of view. For the naive MSE $\mathsf{P}^{\mathrm{N}}_{T+h|T}(\hat{\boldsymbol{\theta}})$, only one set of parameters has to be estimated, and one Kalman filter run has to be computed. For the application of the AK approximation $\mathsf{P}^{\mathrm{AK}}_{T+h|T}(\hat{\boldsymbol{\theta}})$, an additional number of Kalman filter runs equal to the number of parameters must be computed. The PT method on the other hand, relies on B simulations of variables and B iterative ML estimations of the state space model. For thousands of replications and models with a lot of hyperparameters, the relative computational cost is quite high.

3 Monte Carlo evidence

Simulation experiment The setup of the simulation design follows closely Quenneville/Singh (2000), so the results can be compared. The DGP is given by the random walk with noise model

$$
\begin{aligned}
y_t &= \mu_t + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma^2), &(14)\\
\mu_t &= \mu_{t-1} + \eta_t, \qquad \eta_t \sim N(0, \sigma^2 q). &(15)
\end{aligned}
$$

An observed time series y_t is decomposed into a random walk trend component μ_t and a noise component ϵ_t. The random walk is driven by shocks η_t. The disturbances ϵ_t and η_t are mutually uncorrelated and the trend has initial value μ_0. The variance of the noise component is σ^2 and will be concentrated out of the likelihood function. The variance of the permanent random walk shock is equal to $\sigma^2 q$. Because σ^2 is concentrated out, there is only one hyperparameter $\boldsymbol{\theta} = q$ to be estimated in this model. The numerical values for the true parameters are chosen as in Quenneville/Singh (2000): $\mu_0 = 10$, $\sigma^2 = 1$, and $q = 0.25$. The sample size is $T = 50$ and $T = 100$. The DGP is used to replicate artificial 'true' variables $y_{r,t}$ and $\mu_{r,t}$ from (14) and (15) for $r = 1, \ldots, R$ with $R = 1000$. The benchmark for later comparisons is the state at the end of the sample $\mu_{r,T}$. Estimates of the parameter q, denoted as $\hat{q}_r$, are computed applying ML estimation. A diffuse Kalman filter run gives the Kalman filtered state $\hat{\mu}_{r,T|T}(\hat{q}_r)$. Given the simulated true state and the Kalman filtered state, the 'true' MSE can be computed according to

$$
\mathrm{P}^{\mathrm{T}}_{T|T} = \frac{1}{R} \sum_{r=1} \left(\hat{\mu}_{r,T|T}(\hat{q}_r) - \mu_{r,T} \right)^2, \tag{16}
$$

which is computed at the end of the sample T. The term $\hat{\mu}_{r,T|T}(\hat{q}_r)$ is the Kalman filter/smoother of the state at time T, where the parameters are estimated with the data set $y_{b,t}$. $\mathrm{P}^{\mathrm{T}}_{T|T}$ is the benchmark for the alternative MSE approximations naive, AK and PT described before.[10] For each replication r, the alternative MSE approximations are computed and averaged. The average naive MSE, for example, is given by

$$
\mathrm{P}^{\mathrm{N}}_{T|T} = \frac{1}{R} \sum_{r=1} \mathrm{P}^{\mathrm{N}}_{r,T|T}(\hat{q}_r), \tag{17}
$$

[10] We do not compare the MSE described here with the approximation of Hamilton (1986). The results in Quenneville/Singh (2000) show that this approximation is also biased and has a worse performance than the AK method in Monte Carlo simulations.

where $\mathrm{P}^{\mathrm{N}}_{T|T}(\hat{q}_r)$ is the MSE based on the r-th data set $y_{r,t}$ and the estimated hyperparameter $\hat{q}_r$. The next method is the AK approximation

$$\mathrm{P}^{\mathrm{AK}}_{T|T} = \frac{1}{R}\sum_{r=1}^{R}\left[\mathrm{P}^{\mathrm{N}}_{r,T|T}(\hat{q}_r) + \left(\frac{\partial \mu_T}{\partial q}\Big|_{q=\hat{q}_r}\right)' \frac{1}{T} I^{-1}\Big|_{q=\hat{q}_r}\left(\frac{\partial \mu_T}{\partial q}\Big|_{q=\hat{q}_r}\right)\right]_{\{y_t=y_{r,t}\}_1^T}, \tag{18}$$

where the terms in brackets are evaluated numerically for $q = \hat{q}_r$ and based on the data set $y_{r,t}$. Again, an average over all replications gives the average MSE $\mathrm{P}^{\mathrm{AK}}_{T|T}$. For the PT method, each replication r will be used to replicate the model again for $b = 1, \ldots, B$ with $B = 500$. The resulting artificial data is denoted as $y_{b,r,t}$ for $t = 1, \ldots, T$. This artificial data is used to reestimate the model again to obtain $\hat{q}_{b,r}$. The PT MSE approximation is then given by

$$\begin{aligned}\mathrm{P}^{\mathrm{PT}}_{T|T} &= \frac{1}{R}\sum_{r=1}^{R}\left[\frac{1}{B}\sum_{b=1}^{B}\left(\hat{\mu}_{b,r,T|T}(\hat{q}_{r,b}) - \mu_{b,r,T|T}(\hat{q}_r)\right)^2\right.\\ &\quad \left. + \; 2\,\mathrm{P}^{\mathrm{N}}_{r,T|T}(\hat{q}_r) - \frac{1}{B}\sum_{b=1}^{B}\mathrm{P}^{\mathrm{N}}_{b,r,T|T}(\hat{q}_{b,r})\right],\end{aligned} \tag{19}$$

where $\hat{\mu}_{b,r,T|T}(\hat{q}_{r,b})$ is the Kalman filtered estimate of the trend based on the b-th bootstrap replication of data and reestimated parameter $\hat{q}_{b,r}$, and $\mu_{b,r,T|T}(\hat{q}_r)$ is based on the b-th replication of the data $y_{r,t}$ and parameter $\hat{q}_r$. The squared difference of these estimates approximates the estimation uncertainty. The second and the third term on the rhs together provide an estimate of filtering/smoothing uncertainty, where $\mathrm{P}^{\mathrm{N}}_{b,r,T|T}(\hat{q}_{b,r})$ is the naive MSE based on the b-th bootstrap replication and reestimated parameters. The MSE approximation for filtering/smoothing resembles the familiar bootstrap bias correction. A ranking of the various MSEs allows for a judgement about the relative small-sample usefulness of the different approximation methods.

Results The following table (1) shows the true MSE together with the various approximations. The true MSE is for both estimation sample sizes is best approximated by the PT

Table 1: True MSE and approximations

	MSE	
	$T = 50$	$T = 100$
true	0.453	0.427
PT	0.439	0.410
AK	0.418	0.403
naive	0.375	0.383

bootstrap method followed by the AK method. The naive MSE has in both cases the largest bias. When $T = 50$, the bias is around -17%. Therefore, neglecting the uncertainty due to paramete estimation might lead to substantial errors. The uncertainty is considerably underestimated and may lead to wrong policy decisions. With respect to the AK method, the results are consistent with the results by Quenneville/Singh (2000): The AK method improves the bias of the naive MSE considerably. Moreover, the table shows that the bootstrap method might improve the AK method. However, the gains are small.

4 Empirical application to the Euro area

In this section, the PT bootstrap method is applied to forecast the output gap in the Euro area. Therefore, we estimate two bivariate state-space models. The first model is the model proposed by Kuttner (1994), and estimates a model for GDP and inflation. The second one follows closely Rünstler (2002), and includes GDP and capacity utilization. This time series is obtained from surveys in European manufacturing firms, and provides a measure for utilization of the physical capital stock. The models chosen should be seen as examples to introduce and discuss the new bootstrap method of PT. Applications to more sophisticated models are left for future research.

The quarterly data for GDP is taken from the ECB area-wide model database[11] and extended by data from the ECB Monthly Bulletin. The capacity utilization series is kindly provided by Gerhard Rünstler and extended by EU Commission data. The sample range is 1971Q1 to 2002Q4. The estimation sample is up to 2000Q4 (96 observations), and the forecast sample is from 2001Q1 to 2002Q4. Over this forecast horizon of eight quarters, we compute a point forecast of the output gap, as well as the naive MSE and the MSE approximation by PT.

Kuttner model The estimated Kuttner model has the following form

$$y_t = y_t^P + y_t^G \tag{20}$$

$$y_t^P = \underset{(0.00069)}{0.0055} + y_{t-1}^P + \varepsilon_{y^P,t} \qquad \sigma^2_{\varepsilon_{y^P}} = 2.3 \times 10^{-5} \tag{21}$$

$$y_t^G = \underset{(0.10)}{1.81} y_{t-1}^G - \underset{(0.10)}{0.84} y_{t-2}^G + \varepsilon_{y^G,t} \qquad \sigma^2_{\varepsilon_{y^G}} = 1.76 \times 10^{-6} \tag{22}$$

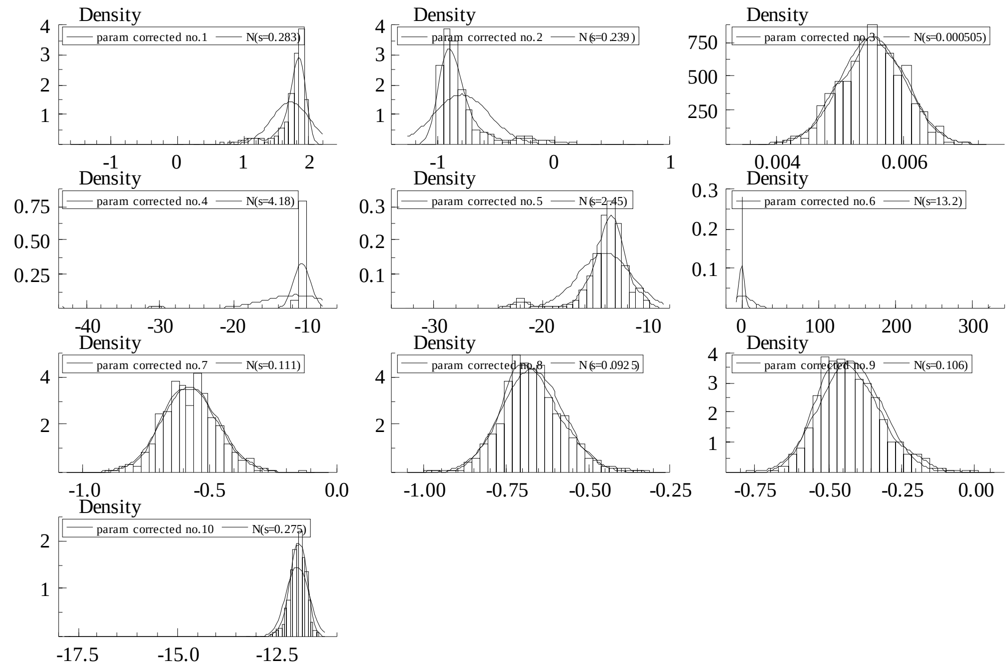
$$\Delta\pi_t = \underset{(0.04)}{0.12} y_{t-1}^G - \underset{(0.12)}{0.53} \Delta\pi_{t-1} - \underset{(0.10)}{0.65} \Delta\pi_{t-2} - \underset{(0.10)}{0.40} \Delta\pi_{t-3} + \varepsilon_{\pi,t} \qquad \sigma^2_{\varepsilon_\pi} = 7.4 \times 10^{-6}, \tag{23}$$

where GDP or output in natural logarithms is denoted as y_t and inflation as π_t. Inflation is calculated as the first differences of the CPI in natural logarithms. Output is decomposed into potential or trend output y_t^P and the output gap y_t^G. Potential output follows a random walk with constant drift υ and is driven by permanent shocks $\varepsilon_{y^P,t}$. The output gap is modelled as an AR(2) process with transitory demand shocks y_t^G. The sum of the autoregressive coefficients is smaller than one, so the output gap is stable. The price equation assumes that prices are I(2) so inflation enters the equation in first differences $\Delta\pi_t$. The log change in inflation is determined by the lagged output gap and three autoregressive terms. Given this fully parameterized model, the Kalman filter can be used to obtain forecasts for the output gap and the corresponding forecast MSE. Moreover, the PT approximation to the true MSE is applied to take into account estimation uncertainty. Therefore, 1000 bootstrap replications of the model according to PT are computed. During the computations, some complications arose. Some of the bootstrap replications led to parameters estimates which implied explosive roots in the output gap or extreme values of the MSE. Therefore, these replications were eliminated for the estimation of the PT MSE.[12] To get an impression about this step, the following figure (1) shows the corrected bootstrap distribution of the model parameters. The numbering of the parameters

[11] See Fagan et al. (2001).

[12] For a similar procedure in a vector autoregressive model context, see Pesaran/Shin (1996).

Figure 1: Bootstrap distribution of parameters in Kuttner model

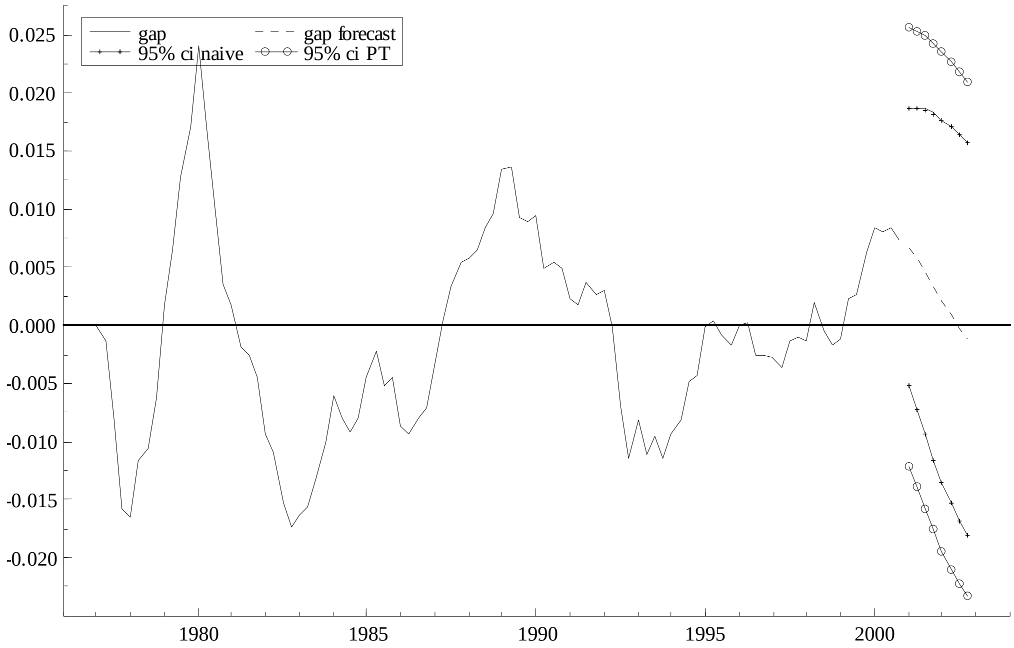


in the graph is as follows

$$y_t^P = \theta_3 + y_{t-1}^P + \varepsilon_{y^P,t},\ \sigma^2_{\varepsilon_{y^P}} = \exp(\theta_4)$$
$$y_t^G = \theta_1 y_{t-1}^G + \theta_2 y_{t-2}^G + \varepsilon_{y^G,t},\ \sigma^2_{\varepsilon_{y^G}} = \exp(\theta_5)$$
$$\Delta\pi_t = \theta_6 y_{t-1}^G + \theta_7 \Delta\pi_{t-1} + \theta_8 \Delta\pi_{t-2} + \theta_9 \Delta\pi_{t-3} + \varepsilon_{\pi,t},\ \sigma^2_{\varepsilon_\pi} = \exp(\theta_{10}).$$

Though some of the most problematic replications have been removed, the corrected bootstrap distribution of the parameters is still skewed with some strong outliers. Except the AR parameters in the inflation equation θ_i for $i = 7, 8, 9$ and the drift parameter θ_3, normality of the parameters can always be rejected. The overall impression is that the estimation uncertainty for this model's parameters is large. Given the replications of the model, the forecast MSE can be obtained. The in-sample output gap together with the output gap forecast and the confidence intervals which correspond to the MSE of the PT method and the naive MSE are displayed in figure (2). The output gap forecast of the Kuttner model is surrounded by a considerable degree of uncertainty. Even without taking into account estimation uncertainty, the output gap forecasts are not significantly different from zero over the whole forecast horizon. The forecast interval obtained with the naive MSE is about 2.5 percentage points wide at forecast horizon one. With estimation uncertainty, the range of the confidence interval increases by approximately 1.0 percentage point on average. Hence, the overall impression is that the forecast uncertainty of the output gap from the Kuttner model is very large so that the information content of these forecasts seems to be quite limited.

Figure 2: Output gap in Kuttner model



Capacity utilization model The capacity utilization model has the form

$$y_t = y_t^P + y_t^G \tag{24}$$

$$y_t^P = \underset{(0.00045)}{0.0056} + y_{t-1}^P + \varepsilon_{y^P,t} \qquad \sigma^2_{\varepsilon_{y^P}} = 1.96 \times 10^{-5} \tag{25}$$

$$y_t^G = \underset{(0.11)}{1.46} y_{t-1}^G - \underset{(0.10)}{0.53} y_{t-2}^G + \varepsilon_{y^G,t} \qquad \sigma^2_{\varepsilon_{y^G}} = 5.56 \times 10^{-6} \tag{26}$$

$$c_t = \underset{(0.43)}{2.34} y_{t-1}^G + \varepsilon_{c,t} \qquad \sigma^2_{\varepsilon_c} = 6.3 \times 10^{-6}, \tag{27}$$

where capacity utilization is denoted as c_t. In this model, a positve comovement between the (lagged) output gap and c_t is assumed. Rünstler (2002) has shown that the inclusion of capacity utilization can reduce estimation and filter uncertainty of the output gap a lot. Again, the Kalman filter can now be used to obtain forecasts for the output gap and the corresponding forecast MSE. To evaluate the estimation uncertainty, 1000 bootstrap replications of the model according to PT are computed. The following figure (3) shows the bootstrap distribution of the model parameters. The numbering of the parameters in the graph is according to

$$y_t^P = \theta_3 + y_{t-1}^P + \varepsilon_{y^P,t},\ \sigma^2_{\varepsilon_{y^P}} = \exp(\theta_4) \tag{28}$$

$$y_t^G = \theta_1 y_{t-1}^G + \theta_2 y_{t-2}^G + \varepsilon_{y^G,t},\ \sigma^2_{\varepsilon_{y^G}} = \exp(\theta_5) \tag{29}$$

$$c_t = \theta_6 y_{t-1}^G + \varepsilon_{c,t},\ \sigma^2_{\varepsilon_c} = \exp(\theta_7). \tag{30}$$

The distribution of the parameters is in most of the cases close to normal. No replication had to be eliminated. Some of the parameters are slightly skewed, and the shock variance of capacity utilization has a strong outlier. However, the bootstrap replications in this model are much more stable than in the Kuttner model. The forecasting results are displayed in figure (4). The measured uncertainty around the output gap is smaller compared to the Kuttner model. In

Figure 3: Bootstrap distribution of parameters in capacity utilization model

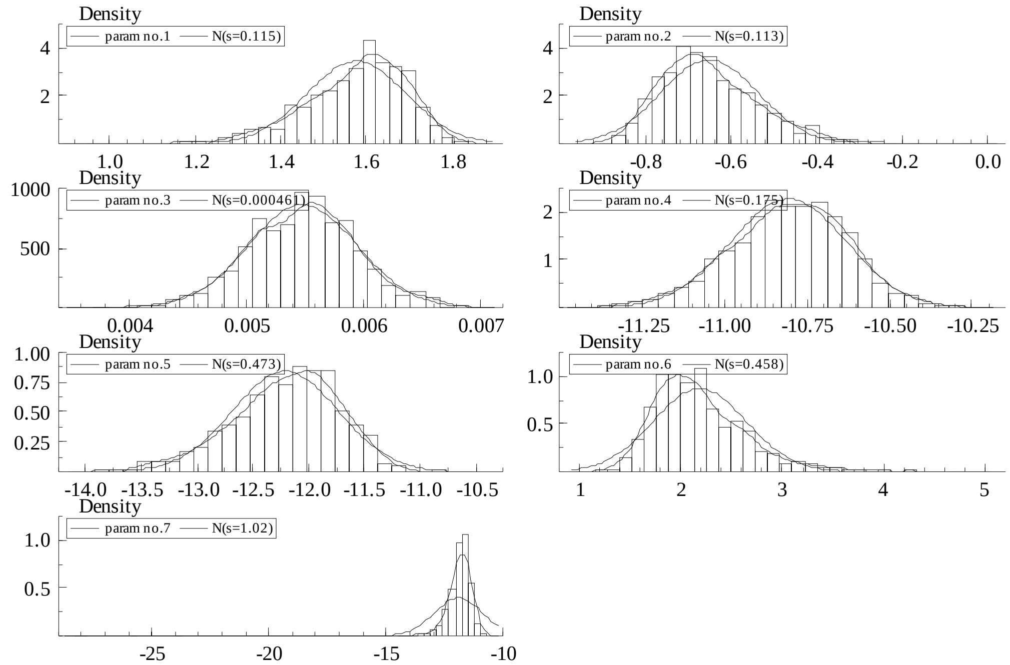
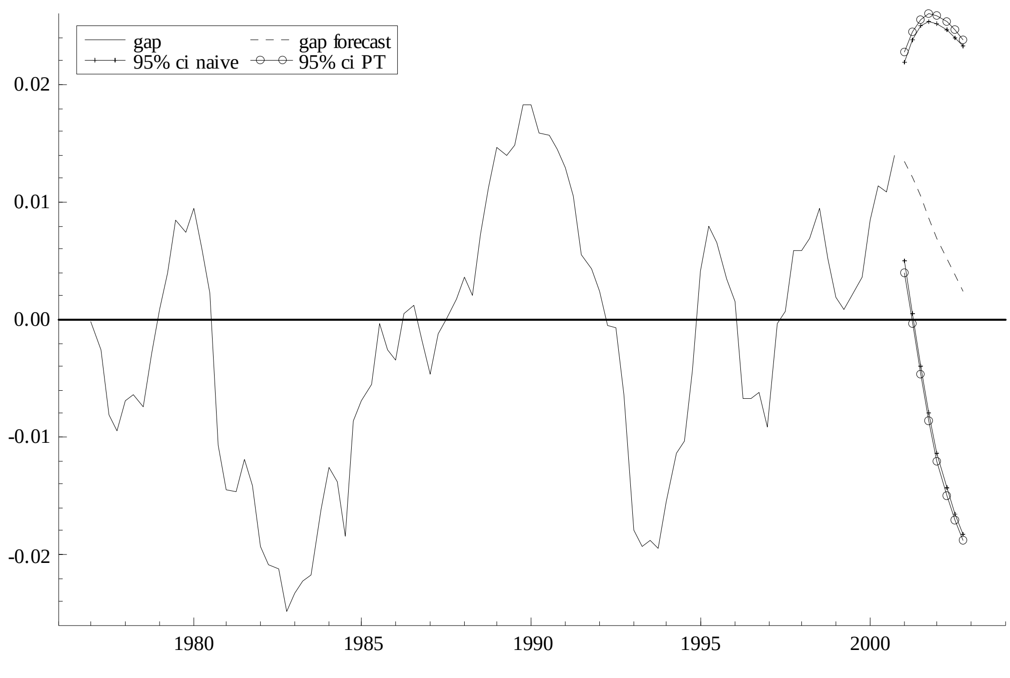


Figure 4: Output gap in capacity utilization model



the first forecasting period, the output gap is significantly larger than zero, and the confidence interval covers around 1.5 percentage points. For forecast horizons two to eight, the confidence interval increases quickly and the output gap is no longer different from zero. Hence, compared to other economic expansions in the Euro area, the out-of-sample information content of this model is limited, too. The PT bootstrap confidence interval is only slightly larger than that based on the naive MSE. Hence, as indicated by the better behaved bootstrap parameters of the model, estimation uncertainty plays only a minor role in this model. Compared with the Kuttner model, however, it must be kept in mind that the capacity utilization model has fewer parameters to estimate. This could be one source of the smaller estimation error. Nonetheless, the overall impression is that capacity utilization together with GDP seems to work better to identify an output gap rather than inflation and GDP.

5 Conclusions

In this paper various methods of MSE approximation in state space models are compared. In a Monte Carlo experiment, a bootstrap method proposed by Pfeffermann/Tiller (2003) works best to reproduce filter and estimation uncertainty around the unobserved components of state space models. The application of this method to predict the output gap with Euro area data shows that estimation uncertainty can play an important role in addition to filter uncertainty. Overall, the forecast intervals around the Euro area output gap are large and indicate a considerable degree of forecast uncertainty. The results obtained so far were derived under the assumption of perfect data availability. However, the recent literature about real-time estimation and forecasting highlights the fact that current data is only preliminary and is often subject to revision. Moreover, publication lags of quarterly data complicates the assessment of the current state of the output gap. Hence, data revisions could be an additional source of uncertainty. Another result of the real-time literature is the non-constant nature of most of the state-space models over time.[13] Hence, structural breaks that occur out-of-sample can induce a change in model parameters and large forecast revisions.

These results show that forecasting the output gap is an extremely difficult task. This is quite unsatisfactory from a practicioner's point of view. If the output gap is hardly distinguishable from zero as the results indicate, inflationary pressures cannot be derived. Hence, the considerable uncertainty questions the usefulness of output gap measures as reliable indicators for monetary policy.

References

[1] Ansley, C., Kohn, R. (1986), Prediction mean squared error for state space models with estimated parameters, Biometrika 73, 467-473.

[2] Camba-Méndez, G., Palenzuela, D. (2003), Assessment criteria for output gap estimates, Economic Modelling.

[3] Clarida, R., Gali, J., Gertler, M. (1999), The Science of Monetary Policy: A New Keynesian Perspective, Journal of Economic Literature 37, 1661-1707.

[4] Doornik, J. (1999), Ox - An object-oriented matrix programming language, Timberlake Consultants.

[13] See Orphanides/Van Norden (2003).

[5] Fagan, G., Henry, J., Mestre, R. (2001), An area-wide model (AWM) for the euro area, ECB Working Paper 42.

[6] Gerlach, S., Smets, F. (1999), Output gaps and monetary policy in the EMU area, European Economic Review 43, 801-812.

[7] Hamilton, J. (1986), A Standard Error for the Estimated State Vector of a State-Space Model, Journal of Econometrics, 33, 387-397.

[8] Hamilton, J. (1994), Time Series Analysis, Princeton University Press.

[9] Harvey, A. (1989), Forecasting, structural time series models and the Kalman filter, Cambridge University Press.

[10] Koopman, S.J., Shephard, N., Doornik, J. (1999), Statistical Algorithms for models in state space using SsfPack 2.2, Econometrics Journal 2, 113-166.

[11] Kuttner, K. (1994), Estimating potential output as a latent variable, Journal of Business and Economic Statistics 12, 361-367.

[12] Orphanides, A., Van Norden, S. (2003), The Unreliability of Output-Gap Estimates in Real-Time, The Review of Economics and Statistics 84, 569-583.

[13] Pesaran, H., Shin Y. (1996), Cointegration and speed of convergence to equilibrium, Journal of Econometrics 71, 117-143.

[14] Pfeffermann, D., Tiller, R. (2003), Bootstrap Approximation to Prediction MSE for State-Space Models with Estimated Parameters, University of Southampton, SSRC Methodology Working Paper M03/05.

[15] Quenneville, B., Singh, A., (2000), Bayesian Prediction Mean Square Error for State Space Models with Estimated Parameters, Journal of Time Series Analysis 21, 219-236.

[16] Rünstler, G. (2002), The information content of real-time output gap estimates, an application to the euro area, ECB Working Paper 182.

[17] Stoffer, D., Wall, K. (1991), Bootstrapping state-space models: Gaussian maximum likelihood estimation and the Kalman filter, Journal of the American Statistical Association 86, 1024-1033.